

# **$w$ -FUNCTION OF THE KDV HIERARCHY**

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**ABSTRACT.** In this paper we construct a family of commuting multidimensional differential operators of order 3, which is closely related to the KdV hierarchy. We find a common eigenfunction of this family and an algebraic relation between these operators. Using these operators we associate a hyperelliptic curve to any solution of the stationary KdV equation. A basic generating function of the solutions of stationary KdV equation is introduced as a special polarization of the equation of the hyperelliptic curve. We also define and discuss the notion of a  $w$ -function of a solution of the stationary  $g$ -KdV equation.

## INTRODUCTION

At the present time various forms of solutions of the stationary  $g$ -KdV equations are known, including the representations with the  $\tau$ -function ([13]),  $\theta$ -function ([14, 15]), and  $\sigma$ -function ([2, 3]); rational solutions can be expressed in terms of Adler-Moser polynomials ([1]). All these functions satisfy the equation

$$(0.1) \quad 2\partial_x^2 \log f = -u,$$

where  $u = u(x, t_2, \dots, t_g)$  is a solution of the stationary  $g$ -KdV equation.

In this paper we construct a family of commuting multidimensional differential operators of third order starting with an arbitrary solution of the stationary  $g$ -KdV equation. Using these operators we solve the following well-known

*Problem 1.* Supplement (0.1) with natural conditions so to obtain a problem with the unique solution.

We call this solution a  $w$ -function of the KdV hierarchy.

In [20] S.P. Novikov observed that each solution of the stationary  $g$ -KdV equation is a  $g$ -gap potential of the Schrödinger operator. It was shown in [2], [3] that the Kleinian  $\sigma$ -function  $\sigma(x, t_2, \dots, t_g)$  provides a solution of the  $g$ -KdV equation. This fact follows from a general result describing all algebraic relations between the higher logarithmic derivatives of the  $\sigma$ -function.

We are going to discuss also the following natural

*Problem 2.* Describe all the relations between the higher logarithmic derivatives

$$\frac{\partial^{i_1 + \dots + i_g}}{\partial x^{i_1} \partial t_2^{i_2} \dots \partial t_g^{i_g}} \log w(x, t_2, \dots, t_g), \quad \text{where } i_1 + \dots + i_g \geq 2$$

following from the construction of the  $w$ -function of the KdV hierarchy.

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A solution of this problem is given in Section 8.

In the paper [16] I.M. Krichever introduced a concept of the Baker–Akhiezer function as a common eigenfunction of the operators  $\mathcal{L}$  and  $A$  (see section 1 for definitions). This function is characterized by its analytic properties, including the behavior at the singular points. In Section 10.2 we express this function via the common eigenfunction of our family of commuting differential operators.

The results of this paper were partially announced in [5], [6].

## 1. PRELIMINARIES

This section is a brief review of basic facts about the KdV hierarchy. See [19] for more details.

The classical KdV (Korteweg – de Vries) equation is

$$(1.1) \quad \frac{\partial}{\partial t} u = \frac{1}{4}(u''' - 6uu'),$$

where  $u$  is a function of real variables  $x$  and  $t$ ; the prime means derivation with respect to  $x$ .

Denote  $\mathcal{L} = \partial_x^2 - u$  the Schrödinger operator with the potential  $u$ . The second term here means *the operator of multiplication* by the function  $u$ ; we will use similar notation throughout the paper. Let also

$$(1.2) \quad A_1 = \partial_x^3 - \frac{3}{4}(u\partial_x + \partial_x u) = \partial_x^3 - \frac{3}{2}u\partial_x - \frac{3}{4}u'.$$

Then, as it was first noticed in [18], the KdV equation is equivalent to the condition

$$[A_1, \mathcal{L}] = \frac{1}{4}(u''' - 6uu').$$

Denote  $\mathfrak{D}$  a ring of differential operators with coefficients in the ring of smooth functions in variables  $x$  and  $t$ . Consider an action of the operator  $\partial/\partial t$  on the ring  $\mathfrak{D}$  defined by the formula

$$(1.3) \quad \frac{\partial}{\partial t} \left( \sum_{k \geq 0} f_k(t, x) \partial_x^k \right) = \sum_{k \geq 0} \frac{\partial f_k(t, x)}{\partial t} \partial_x^k.$$

Then for the operator  $\mathcal{L}$  we obtain the equality

$$\frac{\partial}{\partial t} \mathcal{L} = -\frac{\partial}{\partial t} u.$$

So, equation (1.1) is equivalent to

$$\frac{\partial}{\partial t} \mathcal{L} = [A_1, \mathcal{L}].$$

For every differential operator  $B \in \mathfrak{D}$  define its formal conjugate  $B^*$  as follows: take, by definition,

$$(1.4) \quad \partial_x^* = -\partial_x, \quad f^* = f,$$

where  $f$  is an operator of multiplication by the function  $f$ , and assume  $*$  to be a ring anti-homomorphism:

$$(1.5) \quad (B_1 B_2)^* = B_2^* B_1^*, \quad (B_1 + B_2)^* = B_1^* + B_2^*,$$

for all  $B_1, B_2 \in \mathfrak{D}$ .

We call an operator  $B$  *symmetric* if  $B^* = B$ , and *anti-symmetric* if  $B^* = -B$ . Thus, the operator  $\mathcal{L}$  is symmetric, while the operator  $A_1$  is anti-symmetric.

Consider a subring  $\mathfrak{D}_1 \subset \mathfrak{D}$  generated by  $\partial_x$  and the multiplication operator  $u$ . Supply the ring  $\mathfrak{D}_1$  with a grading such that

$$(1.6) \quad \deg u = 2, \quad \deg \partial_x = 1.$$

Thus,  $\deg u^{(k)} = \deg \partial_x^k u = k + 2$ . The operators  $\mathcal{L}$  and  $A_1$  are then homogeneous of order 2 and 3, respectively.

**Definition 1.1.** Denote by  $\mathfrak{A}$  the linear space of anti-symmetric differential operators  $A$  such that the commutator  $[A, \mathcal{L}]$  is an operator of multiplication by a function.

**Theorem 1.1.** [18] The space  $\mathfrak{A}$  has a basis  $A_0, A_1, \dots$  where  $A_k = \partial_x^{2k+1} + \sum P_{k,i} \partial_x^i$  is a homogeneous differential operator of order  $2k + 1$ , and  $P_{k,i}$  is a differential polynomial of  $u$  of order  $2k + 1 - i$ .

The recurrence relation for the operators  $A_k$  can be found in [8]. The operator  $A_1$  is given by (1.2). The operators  $A_0$  and  $A_2$  are

$$\begin{aligned} A_0 &= \partial_x \\ A_2 &= \partial_x^5 - \frac{5}{4}(u\partial_x^3 + \partial_x^3 u) + \frac{15}{8}u\partial_x u + \frac{5}{16}(u''\partial_x + \partial_x u''). \end{aligned}$$

Denote  $r_k[u] = [A_k, \mathcal{L}]$ , so that  $r_1[u] = \frac{1}{4}(u''' - 6uu')$ ,  $r_2[u] = \frac{1}{16}(u^{(5)} - 10uu''' - 20u'u'' + 30u^2u')$ , etc. Suppose now that  $u$  depends on  $x$  and an infinite set of variables  $t_1, t_2, \dots$ . The equation

$$(1.7) \quad \partial_{t_g} u = r_g[u]$$

is called the  $g$ -th higher KdV equation.

The family of equations (1.7) is called the KdV hierarchy.

The action of differential operators  $\partial_{t_k}$  on the ring  $\mathfrak{D}_u$  is defined similar to (1.3).

**Lemma 1.1.** The operators  $A_k$  satisfy the following “zero curvature” condition:  $\partial_{t_k} A_m - \partial_{t_m} A_k = [A_k, A_m]$ , or, equivalently,  $[\partial_{t_k} - A_k, \partial_{t_m} - A_m] = 0$ .

The expression

$$(1.8) \quad \mathcal{R} = \frac{1}{4}\partial_x^2 - \frac{1}{2}u'\partial_x^{-1} - u,$$

is called the Lenard operator; here  $\partial_x^{-1}$  is an operator of integration with respect to  $x$ . Note that the Lenard operator  $\mathcal{R}$  is multi-valued, to fix its value we need to choose the integration constants.

**Theorem 1.2.** Functions  $r_k[u]$  are related by the Lenard operator:

$$r_{k+1}[u] = \mathcal{R}(r_k[u]).$$

For example,  $r_0 = u'$ , and  $r_1 = \frac{1}{4}u''' - \frac{3}{2}uu' = \frac{1}{4}\partial_x^2 u' - \frac{1}{2}u'u - uu' = \mathcal{R}(r_0)$ .

**Definition 1.2.** The equations

$$(1.9) \quad r_g[u] + \sum_{k=0}^{g-1} a_k r_k[u] = 0,$$

where  $a_k$  are constants, are called higher stationary  $g$ -KdV (or Novikov) equations.

**Theorem 1.3.** *A function  $u$  is a solution of (1.9) if and only if it satisfies the relation  $\mathcal{R}^g(u') = 0$  for some choice of the integration constants; this choice depends on the constants  $a_k$ .*

See [9] for proof.

If one replaces the function  $u$  with  $u + c$  where  $c$  is a constant, then the operator  $A_k$  becomes  $A_k + c \sum_{i=0}^{k-1} c_{k;i} A_i$  for some constants  $c_{k;i}$  where  $c_{k;k-1} \neq 0$ . We can choose the constant  $c$  so that to achieve the equality  $a_{g-1} = 0$  in the decomposition of the operator  $A = A_g + \sum_{k=0}^{g-1} a_k A_k$ .

## 2. A FAMILY OF COMMUTING MULTIDIMENSIONAL DIFFERENTIAL OPERATORS OF ORDER 3

In this section we describe a family  $\{\mathcal{U}_k\}$  of differential operators commuting with each other and with the Schrödinger operator. In this aspect they resemble operators  $\partial_{k+1} - A_k$ , but unlike  $\{A_k\}$  they are multidimensional operators of the third order.

Let  $\{u_1, u_2, \dots, u_g\}$  be a sequence of functions of variables  $t_1 = x, t_2, \dots, t_g$ . Denote  $\partial_i = \partial/\partial t_i$ . Suppose that the first derivatives of the function  $u_1$  are linearly independent, i.e.  $\sum_{i=1}^g c_i \partial_i(u_1) \equiv 0$  only if  $c_1 = \dots = c_g = 0$ . This condition means that the function  $u_1 = u_1(t_1, \dots, t_g)$  essentially depends on all its arguments, i.e. there is no linear projection  $\pi : \mathbb{C}^g \rightarrow \mathbb{C}^{g-1}$  such that  $u = \pi^* \tilde{u}$ , where  $\tilde{u}$  is a function on  $\mathbb{C}^{g-1}$ .

Denote

$$\begin{aligned} \mathcal{L} &= \partial_x^2 - u_1, \\ \mathcal{A}_k &= \partial_x^2 \partial_k - \frac{1}{2}(u_1 \partial_k + \partial_k u_1) - \frac{1}{4}(u_k \partial_x + \partial_x u_k) \end{aligned}$$

where  $k = 1, \dots, g$ . Note that the  $\mathcal{A}_1$  coincides with the operator  $A_1$  given by (1.2).

Define the formal conjugation  $*$  on the space of multidimensional differential operators by formulas (1.5) together with the rule  $\partial_i^* = -\partial_i$  where  $i > 1$ . The operators  $\mathcal{A}_k$  are anti-symmetric:  $\mathcal{A}_k^* = -\mathcal{A}_k$ .

Consider in the ring of differential operators in variables  $t_1, \dots, t_g$  a subring  $\mathfrak{D}_g$  generated by the operators  $\partial_1, \dots, \partial_g$  and  $u_1, \dots, u_g$ . Define on  $\mathfrak{D}_g$  a grading using formulas (1.6) and assuming also that  $\deg u_k = 2k$  and  $\deg \partial_k = 2k - 1$ . It is clear that the operators  $\mathcal{L}$  and  $\mathcal{A}_k$  are homogeneous,  $\deg \mathcal{L} = 2$ ,  $\deg \mathcal{A}_k = 2k + 1$ .

**Lemma 2.1.** *The commutator  $[\mathcal{L}, \mathcal{A}_k]$  is a multiplication operator if and only if  $u'_k = \partial_k u_1$  for all  $k$ . If this condition is satisfied then*

$$(2.1) \quad [\mathcal{L}, \mathcal{A}_k] = \frac{1}{4}(u_k''' - 2u_1' u_k - 4u_1 u_k').$$

**Proof.** Lemma follows from the formula

$$[\mathcal{L}, \mathcal{A}_k] = (-2u_k'' + 2\partial_k u_1')\partial_x + (-u_k' + \partial_k u_1)\partial_x^2 + \partial_k u_1'' - u_1 \partial_k u_1 - \frac{1}{2}u_k u_1' - \frac{3}{4}u_k'''. \quad \square$$

**Lemma 2.2.** *Let  $K$  be an anti-symmetric differential multidimensional operator of order 3. Suppose the commutator  $[K, \mathcal{L}]$  is a multiplication operator, and the coefficients in derivatives of order 3 are constants. Then  $K = \sum_{1 \leq i \leq g} c_i \mathcal{A}_i + \psi_i \partial_i + \phi$ , where  $c_i$  are constants, and the functions  $\psi_i$  and  $\phi$  do not depend on  $x$ .*

**Proof.** Let  $K = \sum_{1 \leq i, j, k \leq g} s_{ijk} \partial_i \partial_j \partial_k + \sum_{1 \leq i, j \leq g} f_{ij} \partial_i \partial_j + \sum_{1 \leq i \leq g} g_i \partial_i + h$ , where  $s_{ijk}$  are constants such that  $s_{ijk} = s_{ikj} = s_{jik}$ , and all the  $f_{ij}$ ,  $g_i$ ,  $h$  are functions of  $t_1, \dots, t_g$ . The anti-symmetry implies that  $f_{ij} = 0$  for all  $i, j$ , and  $\sum_{1 \leq i \leq g} \frac{\partial g_i}{\partial t_i} = 2h$ .

We have

$$[\mathcal{L}, K] = \sum_{1 \leq i \leq g} g_i'' \partial_i + 2 \sum_{1 \leq i \leq g} g_i' \partial_x \partial_i + h'' + 2h' \partial_x + \sum_{1 \leq i \leq g} g_i \frac{\partial}{\partial t_i} u_1 + \sum_{1 \leq i, j, k \leq g} s_{ijk} \left( \frac{\partial^3}{\partial t_i \partial t_j \partial t_k} u_1 + 3 \frac{\partial^2}{\partial t_i \partial t_j} u_1 \partial_k + 3 \frac{\partial}{\partial t_i} u_1 \partial_j \partial_k \right)$$

Since the commutator  $[\mathcal{L}, K]$  is a multiplication operator, the coefficients at  $\partial_i \partial_j$  and  $\partial_i$  in the last formula are zeros.

It follows from the linear independence for the first derivatives of the function  $u_1$  that  $s_{ijk} = 0$  when  $i, j \neq 1$ . If  $i \neq 1$  then one has  $2g_i' = -3s_{11i} \frac{\partial}{\partial x} u_1$ .

Equalizing the coefficient at  $\partial_x$  to zero, we obtain  $g_1'' + 2h' + 3 \sum_{1 \leq i \leq g} s_{11i} \frac{\partial^2 u}{\partial_i \partial x} = 0$ . Put  $c_i = 3s_{11i} = s_{11i} + s_{1i1} + s_{i11}$ ;  $\psi_1 = g_1 + 1/2 \sum_{i=1}^g c_i u_i$ ,  $\psi_k = g_k + \sum_{i=1}^g c_i u_i$  for  $k \neq 1$ , and  $\phi = h + \sum_{i=1}^g c_i u_i'$ . Then the functions  $\psi_i$ ,  $i = 1, \dots, g$  and  $\phi$  do not depend on  $x$  and  $K = \sum_{1 \leq i \leq g} c_i \mathcal{A}_i + \psi_i \partial_i + \phi$ .  $\square$

Denote

$$\begin{aligned} \mathcal{U}_i &= \mathcal{A}_i - \partial_{i+1}, \quad \text{for } i < g; \\ \mathcal{U}_g &= \mathcal{A}_g. \end{aligned}$$

The operators  $\mathcal{U}_i$  are anti-symmetric and homogeneous.

**Lemma 2.3.** *The following conditions are equivalent*

- (1)  $[\mathcal{L}, \mathcal{U}_k] = 0$ .
- (2)  $-\partial_{k+1} \mathcal{L} = [\mathcal{L}, \mathcal{A}_k]$ .
- (3)  $\partial_{k+1} u_1 = u_{k+1}' = \frac{1}{4}(u_k''' - 2u_1' u_k - 4u_1 u_k')$  for  $k < g$ , and  $(u_g''' - 2u_1' u_g - 4u_1 u_g') = 0$ .

**Proof.** Lemma follows from the equality

$$[\mathcal{L}, \mathcal{U}_k] = [\mathcal{L}, \mathcal{A}_k] + \partial_{k+1} \mathcal{L} = [\mathcal{L}, \mathcal{A}_k] - \partial_{k+1} u_1. \quad \square$$

The last statement in Lemma 2.3 allows to express the functions  $u_k$  recursively via  $u_1$  and its  $x$ -derivative, up to the choice of a function that do not depend on  $x$ .

**Corollary 2.1.** *Under the hypotheses of Lemma 2.3, functions  $u_i$  are related by the Lenard operator  $\mathcal{R}$  (see (1.8)):*

$$(2.2) \quad \partial_{i+1} u_1 = \mathcal{R}(u_i') = \mathcal{R}^{i+1}(u_1').$$

For  $i = g$  one has  $0 = \partial_{g+1} u_1 = \mathcal{R}^g(u_1')$ .

**Corollary 2.2.** *The operators  $\{\mathcal{U}_k\}$  commute with  $\mathcal{L}$  if and only if the function  $u_1(x)$  is a solution of the stationary  $g$ -KdV equation.*

**Lemma 2.4.** *The operators  $\mathcal{U}_i, \mathcal{U}_j$  commute for all  $1 \leq i, j \leq g$  if and only if the functions  $\{u_i\}$  satisfy condition (3) of Lemma 2.3 and the following equalities:*

$$(2.3) \quad \partial_i u_j = \partial_j u_i,$$

$$(2.4) \quad u_k' u_i - u_i' u_k + 2\partial_{i+1} u_k - 2\partial_{k+1} u_i = 0, \quad 1 \leq i, k \leq g.$$

Lemma is proved by direct calculation.

Note that (2.3) implies the existence of a function  $z(t_1, \dots, t_g)$  such that  $\partial_i z = u_i$ .

The commutation of operators  $\mathcal{U}_i$  is equivalent to zero curvature conditions for the operators  $\mathcal{A}_i$ :

$$(2.5) \quad \partial_{j+1} \mathcal{A}_i - \partial_{i+1} \mathcal{A}_j + [\mathcal{A}_i, \mathcal{A}_j] = 0$$

### 3. A GENERALIZED TRANSLATION ASSOCIATED WITH THE KdV HIERARCHY

In this section we develop the technique of a generalized translation from [3].

For  $\eta \in \mathbb{R}$  define an operator  $D^\eta$  acting on the space of functions of one variable as  $(D^\eta f)(\xi) = \frac{\xi\eta}{2(\xi-\eta)} f(\eta)$ . Define the operator  $\mathcal{B}$  by the rule

$$\mathcal{B}(f, h)(\xi, \eta) = \frac{\xi\eta}{2(\xi-\eta)} (f(\xi)h(\eta) - f(\eta)h(\xi)) = f(\xi)(D^\eta h)(\xi) - g(\xi)(D^\eta f)(\xi).$$

It possesses the following properties:

- $\mathcal{B}(f, h)(\xi, \eta) = -\mathcal{B}(h, f)(\xi, \eta)$ .
- $\mathcal{B}(f, h)(\xi, \eta) = \mathcal{B}(f, h)(\eta, \xi)$ .
- $\mathcal{B}$  is a bilinear operator.
- If  $f(\xi)$  and  $h(\xi)$  are polynomials, then  $\mathcal{B}(f, h)(\xi, \eta)$  is also a polynomial.
- $\mathcal{B}(f, \xi^{-1})(\xi, \eta) = \frac{f(\xi)\xi - f(\eta)\eta}{2(\xi - \eta)}$ .
- $\mathcal{B}(1, 2\xi^{-1}) = 1$ .

Define also an operator  $\mathcal{B}_k$  acting on the set of  $k$ -tuples of functions of one variable as follows:

$$\begin{aligned} \mathcal{B}_k(f_1, \dots, f_k)(\xi_1, \dots, \xi_k) &= \frac{\prod_{i=1}^k \xi_i^{k-1}}{2^{k-1} \prod_{1 \leq i < j \leq k} (\xi_i - \xi_j)} \left( \sum_{\sigma \in \mathcal{S}_k} (-1)^\sigma f_1(\xi_{\sigma(1)}) \dots f_k(\xi_{\sigma(k)}) \right) = \\ &= \frac{\prod_{i=1}^k \xi_i^{k-1}}{2^{k-1} W(\xi_1, \xi_2, \dots, \xi_k)} \begin{vmatrix} f_1(\xi_1) & f_2(\xi_1) & \dots & f_k(\xi_1) \\ f_1(\xi_2) & f_2(\xi_2) & \dots & f_k(\xi_2) \\ \vdots & \vdots & & \vdots \\ f_1(\xi_k) & f_2(\xi_k) & \dots & f_k(\xi_k) \end{vmatrix} \end{aligned}$$

where  $W(\xi_1, \dots, \xi_k)$  is the Vandermonde determinant.

Note that  $\mathcal{B}_1(f) = f$ ,  $\mathcal{B}_2(f, g) = \mathcal{B}(f, g)$ .

Let  $f_i$  be a function of variables  $\xi; t_1, \dots, t_g$ . So that one has

$$\partial_j \mathcal{B}_k(f_1, \dots, f_k)(\xi_1, \dots, \xi_k) = \sum_{i=1}^k \mathcal{B}_k(f_1, \dots, \partial_j f_i, \dots, f_k)(\xi_1, \dots, \xi_k).$$

For a fixed function  $h$  put, by definition,  $(T_\xi^\eta f)(\xi) = (T(h)_\xi^\eta f)(\xi) = \mathcal{B}(f, h)(\xi, \eta)$ .

**Lemma 3.1.** *The operators  $T_\xi^\eta$  satisfy the associativity condition  $T_\xi^\eta T_\xi^\tau = T_\eta^\tau T_\xi^\eta$  and the commutativity condition  $T_\xi^\eta T_\xi^\tau = T_\xi^\tau T_\xi^\eta$ .*

**Proof.** Calculate  $T_\xi^\tau T_\xi^\eta f$ :

$$\begin{aligned} T_\xi^\tau T_\xi^\eta f &= \frac{\xi\tau}{2(\xi-\tau)} \left( \xi\eta \frac{f(\xi)h(\eta) - f(\eta)h(\xi)}{2(\xi-\eta)} h(\tau) - \tau\eta \frac{f(\tau)h(\eta) - f(\eta)h(\tau)}{2(\tau-\eta)} h(\xi) \right) = \\ &= \frac{\xi^2\eta^2\tau^2}{4(\xi-\tau)(\xi-\eta)(\eta-\tau)} \frac{f(\xi)h(\eta)h(\tau)(\tau^{-1} - \eta^{-1}) + f(\eta)h(\tau)h(\xi)(\xi^{-1} - \tau^{-1}) + f(\tau)h(\xi)h(\eta)(\eta^{-1} - \xi^{-1})}{1} = \\ &= \mathcal{B}_3(f(\xi), h(\xi), h(\xi)\xi^{-1})(\xi, \eta, \tau). \end{aligned}$$

This expression is invariant under all the permutations of the variables  $\xi, \eta, \tau$ . Lemma is proved.  $\square$

**Corollary 3.1.** *The operator  $T_\xi^\eta$  is a commutative operator of generalized translation and*

$$T_\xi^\eta 1 = \frac{\xi\eta}{2(\xi-\eta)} (h(\eta) - h(\xi)).$$

In particular,  $T_\xi^\eta 1 = 1$  if and only if  $h(\xi) = 2/\xi$ .

*Remark 3.1.* The generalized translation operator  $\mathcal{D}_\xi^\eta(f) = \frac{\xi f(\xi) - \eta f(\eta)}{\xi - \eta}$  from [3] is equal to  $T_\xi^\eta$  when  $h = 2/\xi$ .

*Remark 3.2.* Let  $h(\xi) = h_{-1}/\xi + \tilde{h}(\xi)$  where  $\tilde{h}(\xi)$  is a function regular in a neighbourhood of the origin. Then for a function  $f(\xi)$  regular in the vicinity of the origin the function  $f(\xi, \eta) = T_\xi^\eta f$  is regular in the vicinity of the point  $(\xi, \eta) = (0, 0)$ .

**Definition 3.1.** *A polarization of a smooth function  $f(\xi)$  is a symmetric function of two variables  $f(\xi, \eta)$  such that  $f(\xi, \eta) = 2f(\xi)$ .*

**Lemma 3.2.** *Let  $f(\xi, \eta)$  be a polarization of a function  $f(\xi)$ . Then*

$$(3.1) \quad \left. \frac{\partial f(\xi, \eta)}{\partial \xi} \right|_{\xi=\eta} = \frac{\partial f(\xi)}{\partial \xi}.$$

**Proof.** For a symmetric function  $f(\xi, \eta)$  there exists a function  $h(s_1, s_2)$  such that  $f(\xi, \eta) = h(\xi + \eta, \xi\eta)$ . Since  $2f(\xi) = f(\xi, \xi) = h(2\xi, \xi^2)$ , one has

$$\left. \frac{\partial f(\xi, \eta)}{\partial \xi} \right|_{\xi=\eta} = \frac{\partial h}{\partial s_1}(\xi + \eta, \xi\eta) + \frac{\partial h}{\partial s_2}(\xi + \eta, \xi\eta)\eta \Big|_{\xi=\eta} = \frac{\partial h}{\partial s_1}(2\xi, \xi^2) + \frac{\partial h}{\partial s_2}(2\xi, \xi^2)\xi.$$

On the other hand,

$$\frac{\partial f}{\partial \xi}(\xi) = \frac{1}{2} \frac{\partial h(2\xi, \xi^2)}{\partial \xi} = \frac{\partial h}{\partial s_1}(2\xi, \xi^2) + \frac{\partial h}{\partial s_2}(2\xi, \xi^2)\xi = \left. \frac{\partial f(\xi, \eta)}{\partial \xi} \right|_{\xi=\eta} \quad \square$$

**Example 1.** Let  $f(\xi) = \sum_i g_i(\xi)h_i(\xi)$ . Then the function  $f(\xi, \eta) = \sum_i (g_i(\xi)h_i(\eta) + g_i(\eta)h_i(\xi))$  is the polarization of  $f(\xi)$ .

Let  $F_n$  is a set of smooth functions on  $n$  variables.

**Definition 3.2.** Let  $G : F_1^k \rightarrow F_1$  and  $\widehat{G} : F_1^k \rightarrow F_2$ . The operator  $\widehat{G}$  is called a polarization of the operator  $G$  if the function  $\widehat{G}(f_1, \dots, f_k)$  is a polarization of the function  $G(f_1, \dots, f_k)$  for any  $f_1, \dots, f_k$ .

Recall ([13]) that the one-variable Hirota operator  $H_\xi$  is given by

$$H_\xi[f(\xi), g(\xi)] = f'(\xi)g(\xi) - f(\xi)g'(\xi).$$

**Lemma 3.3.** *The operator  $\frac{2}{\xi\eta}\mathcal{B}(f, g)(\xi, \eta)$  gives the polarization of the Hirota operator*

**Proof.** We need to prove that

$$\lim_{\eta \rightarrow \xi} \mathcal{B}(f, g)(\xi, \eta) = \frac{\xi^2}{2} \mathcal{H}_\xi[f(\xi), g(\xi)].$$

Let  $\eta = \xi + \varepsilon$ . Then  $f(\eta) = f(\xi) + \varepsilon f'(\xi) + O(\varepsilon^2)$  and  $g(\eta) = g(\xi) + \varepsilon g'(\xi) + O(\varepsilon^2)$ . Hence  $\mathcal{B}(f, g)(\xi, \xi + \varepsilon) = \frac{\xi(\xi + \varepsilon)}{-2\varepsilon} (f(\xi)g'(\xi)\varepsilon - g(\xi)f'(\xi)\varepsilon + O(\varepsilon^2)) = \frac{\xi^2}{2} \mathcal{H}_\xi[f(\xi), g(\xi)] + O(\varepsilon)$   $\square$

Define operators  $D_i$  with the help of the expansion

$$(D^\eta f)(\xi) = \sum_{i \in \mathbb{Z}} (D_i f)(\xi) \eta^i.$$

**Lemma 3.4.** *Let  $f(\xi) = \dots + f_0 + f_1\xi + f_2\xi^2 + \dots$ . Then*

$$(D_k f)(\xi) = \frac{1}{2}(\dots + f_0\xi^{-k+1} + \dots + f_{k-1}).$$

*If  $f(\xi) = f_0 + f_1\xi + \dots + f_n\xi^n$  is a polynomial then  $(D_1 f)(\xi) = \frac{1}{2}f_0$  and  $(D_{n+1} f)(\xi) = \frac{1}{2}\xi^{-n}f(\xi)$ .*

It is clear that

$$(3.2) \quad (D_{k+1} f)(\xi) = \xi^{-1}(D_k f)(\xi) + \frac{1}{2}f_k.$$

Note one more property of the operators  $D_i$ :

**Lemma 3.5.** *Let  $f(\xi)$  be a polynomial. Then*

$$\sum_{k \geq 0, m \geq 0} D_{k+m+1}(f(\xi)) \eta^k \zeta^m = \frac{\xi}{2(\eta - \zeta)} \left( \frac{\eta}{\xi - \eta} f(\eta) - \frac{\zeta}{\xi - \zeta} f(\zeta) \right) = \frac{2}{\eta \zeta} \mathcal{B} \left( 1, D_\xi^\eta f(\xi) \right) (\eta, \zeta).$$

Define the operators  $d_i$  by the formula  $T_\xi^\eta f(\xi) = \sum (d_i f(\xi)) \eta^i$ . Then

$$d_i f(\xi) = f(\xi) D_i h(\xi) - h(\xi) D_i f(\xi).$$

From Lemma 3.1 using the standard methods we obtain:

**Lemma 3.6.** *The linear space spanned by the operators  $d_i$ ,  $i = 1, \dots$ , is an associative and commutative algebra with the following multiplication:*

$$d_i d_j = \sum_{k=0}^{i+j} c_{ij}^k d_k$$

where the structure constants  $c_{ij}^k$  are found from the expansion

$$T_\xi^\eta \xi^k = \frac{\xi \eta}{2(\xi - \eta)} (\xi^k h(\eta) - \eta^k h(\xi)) = \sum_{i+j \geq k} c_{ij}^k \xi^i \eta^j.$$

For the sequence of function  $\{u_1, \dots, u_g\}$  of variables  $t_1 = x, t_2, \dots, t_g$  introduce the generating functions

$$\mathbf{u}(\xi) = \sum_{i=1}^g u_i \xi^i, \quad \mathbf{u}'(\xi) = \sum_{i=1}^g u'_i \xi^i, \quad \dots, \quad \mathbf{u}^{(k)}(\xi) = \sum_{i=1}^g u_i^{(k)} \xi^i$$



(prime here, as usual, means a differentiation with respect to  $x$ ). The following statement gives an expression of the third derivatives  $u_1''', \dots, u_g'''$  in terms of functions  $u_1, \dots, u_g$  and their first derivatives. Moreover, it allows to express these derivatives recursively as a differential polynomial in  $u_1$ . This is one of the key results of the paper:

**Theorem 3.1.** *The sequence  $\{u_1, u_2, \dots, u_g\}$  satisfies condition (3) of Lemma 2.3 if and only if the generating function  $\mathbf{u}(\xi)$  is a solution of the following equation:*

$$(3.3) \quad \mathbf{u}'''(\xi) + 2u_1'(2 - \mathbf{u}(\xi)) - 4(\xi^{-1} + u_1) \mathbf{u}'(\xi) = 0.$$

**Proof.** We have

$$\mathbf{u}'''(\xi) + 2u_1'(2 - \mathbf{u}(\xi)) - 4(\xi^{-1} + u_1) \mathbf{u}'(\xi) = \sum_{i=1}^g (u_i''' - 2u_1' u_i - 4u_1 u_i' - 4u_{i+1}') \xi^i.$$

The coefficients at  $\xi^i$  in the right-hand side of this formula are all zero if and only if condition (3) of Lemma 2.3 holds.  $\square$

Take, by definition,

$$\partial_k \mathbf{u}(\xi) = \sum_{i=1}^g \partial_k u_i \xi^i.$$

**Lemma 3.7.** *Equations (2.3) and (2.4) together are equivalent to the following equation:*

$$\partial_{k+1} \mathbf{u}(\xi) = \xi^{-1} \partial_k \mathbf{u}(\xi) - \frac{1}{2} u_k \mathbf{u}'(\xi) + \frac{1}{2} u_k' (\mathbf{u}(\xi)) - u_k'.$$

*This equation allows to determine recursively the partial derivatives  $\partial_k \mathbf{u}(\xi)$ :*

$$(3.4) \quad \partial_k \mathbf{u}(\xi) = D_k(2 - \mathbf{u}(\xi)) \mathbf{u}'(\xi) - D_k(\mathbf{u}'(\xi))(2 - \mathbf{u}(\xi)).$$

In the sequel we suppose that (3.3) and (3.4) hold for the function  $\mathbf{u}(\xi)$ .

**Corollary 3.2.**

$$(3.5) \quad \partial_k \mathbf{u}'(\xi) = D_k(2 - \mathbf{u}(\xi)) \mathbf{u}''(\xi) - D_k(\mathbf{u}''(\xi))(2 - \mathbf{u}(\xi)),$$

$$(3.6) \quad \begin{aligned} \partial_k \mathbf{u}''(\xi) &= 4(\xi^{-1} + u) \partial_k \mathbf{u}(\xi) - 2u_k'(2 - \mathbf{u}(\xi)) \\ &\quad + D_k(\mathbf{u}''(\xi)) \mathbf{u}'(\xi) - D_k(\mathbf{u}'(\xi)) \mathbf{u}''(\xi). \end{aligned}$$

Let  $\partial(\eta) = \sum_{i=1}^g \eta^i \partial_i$ . Note that for a fixed  $\eta$  the operator  $\partial(\eta)$  is an operator of differentiation in the direction of the vector  $(\eta, \eta^2, \dots, \eta^g)$ , i.e.

$$\partial(\eta) f(t_1, \dots, t_g) = \left. \frac{\partial}{\partial \tau} f(t_1 + \tau \eta, \dots, t_g + \tau \eta^g) \right|_{\tau=0}.$$

**Corollary 3.3.**

$$(\partial_x^2 \partial(\xi) + 2(2 - \mathbf{u}(\xi)) \partial_x - 4(\xi^{-1} + u_1) \partial(\xi)) u = 0.$$

**Proof.** Recall that  $\partial_i u_1 = u_i'$ , and therefore  $\mathbf{u}'(\xi) = \partial(\xi) u_1$  and  $\mathbf{u}'''(\xi) = \partial_x^2 \partial(\xi) u$ . The statement follows now from (3.3).  $\square$

Denote  $\mathcal{T}_\xi^\eta = T(2 - \mathbf{u}(\xi))_\xi^\eta$ . The operator  $\mathcal{T}_\xi^\eta$  plays a special role below, as shows

**Theorem 3.2.**

$$(3.7) \quad \partial(\eta) \mathbf{u}(\xi) = \mathcal{T}_\xi^\eta \partial_x \mathbf{u}(\xi).$$

$$(3.8) \quad \partial(\eta) \mathbf{u}'(\xi) = \mathcal{T}_\xi^\eta \partial_x^2 \mathbf{u}(\xi).$$

$$(3.9) \quad \partial(\eta) \mathbf{u}''(\xi) = \mathcal{T}_\xi^\eta \partial_x^3 \mathbf{u}(\xi) - \mathcal{B}_\xi^\eta(\mathbf{u}'(\xi), \mathbf{u}''(\xi)).$$

**Proof.** These formulas follow from the definition of the operator  $\mathcal{T}_\xi^\eta$  and equations (3.4), (3.5), (3.6).  $\square$

Note that (3.7), (3.8) imply that

$$[\partial_x, \mathcal{T}_\xi^\eta] \partial_x \mathbf{u}(\xi) = 0.$$

The associativity condition for the operator  $\mathcal{T}_\xi^\eta$  is equivalent to the following relation, which will be used later:

$$(3.10) \quad \partial(\zeta) \frac{\xi \eta}{2(\xi - \eta)} \frac{2 - \mathbf{u}(\eta)}{2 - \mathbf{u}(\xi)} = \frac{1}{(2 - \mathbf{u}(\xi))^2} \mathcal{T}_\xi^\eta \mathcal{T}_\xi^\zeta \mathbf{u}'(\xi).$$

Now we describe the family of differential operators  $\{\mathcal{U}_i\}$  using the method of generating function.

**Lemma 3.8.** *The generating function of the sequence of operators  $\mathcal{U}_i$  is:*

$$\sum_{i=1}^g \mathcal{U}_i \xi^i = \frac{1}{2} ((\mathcal{L} - \xi^{-1}) \partial(\xi) + \partial(\xi) (\mathcal{L} - \xi^{-1})) + \frac{1}{4} ((2 - \mathbf{u}(\xi)) \partial_x + \partial_x (2 - \mathbf{u}(\xi))).$$

#### 4. THE HYPERELLIPTIC CURVE ASSOCIATED WITH A SOLUTION OF KdV

**Theorem 4.1.** *Suppose the generating function  $\mathbf{u}(\xi)$  satisfies (3.3) and (3.4). Let*

$$(4.1) \quad 4\mu(\xi) = \mathbf{u}'(\xi)^2 + 2\mathbf{u}''(\xi) (2 - \mathbf{u}(\xi)) + 4(\xi^{-1} + u_1) (2 - \mathbf{u}(\xi))^2.$$

*Then  $\mu(\xi) = 4\xi^{-1} + \sum_{i=1}^{2g} \mu_i \xi^i$  where  $\mu_i$  are constants,  $i = 1, \dots, 2g$ .*

**Proof.** It follows from (3.7), (3.8), (3.9) that  $\partial(\eta)\mu(\xi) = 2\mathbf{u}'(\xi)\partial(\eta)\mathbf{u}'(\xi) + 2(2 - \mathbf{u}(\xi))\partial(\eta)\mathbf{u}''(\xi) - 2\mathbf{u}''(\xi)\partial(\eta)\mathbf{u}(\xi) + 4\mathbf{u}'(\eta)(2 - \mathbf{u}(\xi))^2 - 8(\xi^{-1} + u_1)(2 - \mathbf{u}(\xi))\partial(\eta)\mathbf{u}(\xi) = 0$ . Therefore  $\partial_i \mu_j = 0$  where  $1 \leq i \leq g$ ,  $1 \leq j \leq 2g$ , and all the  $\mu_i$  are constants.  $\square$

Assume  $u_k = 0$  for  $k > g$ . Equation (4.1) implies that

$$(4.2) \quad u_{k+1} = \frac{1}{4} \mu_k + J_k(u, u', u'', \dots, u_k, u'_k, u''_k), \quad k = 1, \dots, 2g,$$

where  $J_k$  are polynomials. We see that the functions  $u_k$ ,  $k = 2, \dots, g$ , can be expressed recursively via the function  $u_1$ , its derivatives and the constants  $\mu_i$ , namely

$$(4.3) \quad u_k = \Theta_k(u, u', \dots, u^{(2k-2)}, \mu_1, \dots, \mu_{k-1})$$

where  $\Theta_k$  are polynomials.

Note that the condition  $J'_g = 0$  is equivalent to the stationary  $g$ -KdV equation. For  $k > g$  one has  $J_k = -1/4\mu_k$ , which gives rise to integrals of the higher KdV equation (see [19]).

Since  $\partial_k u_1 = u'_k$ , the partial derivative of  $u_1$  with respect to  $t_k$  can also be expressed in terms of derivatives with respect to  $x$ . Therefore the behavior of function  $u_1$  along the coordinate axes  $t_2, \dots, t_g$  can be reconstructed if its derivatives with respect to  $x$  are known.

**Lemma 4.1.** *Let  $\mu_i$  be constants. Then equation (4.1) implies equation (3.3). If equations (2.3) and (4.1) hold, then equation (3.4) holds, too.*

**Proof.** The first statement of the lemma is trivial. From the equality  $u'_{k+1} = 1/4(u'''_k - 2u'u_k - 4uu'_k)$  one obtains

$$\partial_{k+1}u'_m = \partial_m u'_{k+1} = 1/4(\partial_m u'''_k - 2u''_m u_k - 2u' \partial_m u_k - 4u'_m u'_k - 4u \partial_m u'_k) = \partial_k u'_{m+1} + 1/2 u''_k u_m - 1/2 u''_m u_k.$$

This equation proves that (3.5) holds. So (3.6) is also true. Integration of (3.5) with respect to  $x$  gives the formula  $\partial_k \mathbf{u}(\xi) = D_k(2 - \mathbf{u}(\xi))\mathbf{u}'(\xi) - D_k(\mathbf{u}'(\xi))(2 - \mathbf{u}(\xi)) + \varphi(t_2, \dots, t_g)$  where the function  $\varphi$  does not depend on  $x$ . Combining the last equation with (4.1), we obtain

$$0 = 4 \partial_k \mu(\xi) = \partial_k (\mathbf{u}'(\xi)^2 + 2\mathbf{u}''(\xi)(2 - \mathbf{u}(\xi)) + 4(\xi^{-1} + u_1)(2 - \mathbf{u}(\xi))^2) = (8(\xi^{-1} + u_1)(\mathbf{u}(\xi)) + 2\mathbf{u}''(\xi)) \varphi(t_2, \dots, t_g).$$

The function in parenthesis cannot vanish identically as a function of  $x$ , and thus  $\varphi(t_2, \dots, t_g) \equiv 0$ .  $\square$

Summarize the results obtained:

**Theorem 4.2.** *The following statements are equivalent:*

- (1) *The function  $u_1$  is a solution of a stationary  $g$ -KdV equation.*
- (2) *There exists a sequence of functions  $\{u_1, \dots, u_g\}$  such that the operators  $\mathcal{L} = \partial_x^2 - u_1$  and  $\mathcal{U}_i = \partial_x^2 \partial_i - \frac{1}{2}(u_1 \partial_i + \partial_i u_1) - \frac{1}{4}(u_i \partial_x + \partial_x u_i) - \partial_{i+1}$ ,  $1 \leq i \leq g$  commute.*
- (3) *There exists a sequence of functions  $\{u_1, \dots, u_g\}$  and a set of constants  $\mu_1, \dots, \mu_{2g}$  such that the generating function  $\mathbf{u}(\xi) = \sum_{i=1}^g u_i \xi^i$  satisfies (4.1).*

In order to find out the relation between the constants  $\mu_k$  and the coefficients  $a_i$  we need the following result:

**Lemma 4.2.** *The operator  $\mathbf{U} = \mathcal{U}_1 \mathcal{L}^{g-1} + \mathcal{U}_2 \mathcal{L}^{g-2} + \dots + \mathcal{U}_g$*

- (1) *commutes with the operator  $\mathcal{L}$ ;*
- (2) *is an operator of order  $2g + 2$  with the leading coefficient 1;*
- (3) *contains the differentiation with respect to  $x$  only.*

**Proof.** The first statement of the lemma is obvious. The leading term of  $\mathbf{U}$  is a composition of the leading terms of the operators  $\mathcal{U}_1$  and  $\mathcal{L}^{g-1}$ , so it is equal to  $\partial_x^{2g+2}$ . This proves the second statement. Since  $\mathcal{U}_i = \partial_i \mathcal{L} - \partial_{i+1} - 1/2 u_i \partial_x + 1/4 u'_i$ , the sum  $\mathcal{U}_i \mathcal{L} - \mathcal{U}_{i+1}$  does not contain the differentiation with respect to  $t_{i+1}$ . By recursion we get the third statement of the lemma.  $\square$

**Theorem 4.3.** *Under the hypotheses of Theorem 4.2 the operator  $A$  can be decomposed as  $A = \mathcal{U}_1 \mathcal{L}^{g-1} + \mathcal{U}_2 \mathcal{L}^{g-2} + \dots + \mathcal{U}_g$ , where  $A = A_g + \sum_{i=0}^{g-2} a_i A_i$ . The coefficients  $\mu_k$  and  $a_i$  satisfy the following relation:*

$$(4.4) \quad \mu_k = 8a_{g-k-1} + 4 \sum_{i=1}^{k-2} a_{g-i-1} a_{g-k+i}.$$

for  $k = 1, \dots, g-1$ .

**Proof.** The first statement of the theorem follows from lemma 4.2 and uniqueness of the operators  $A_k$  (see Theorem 1.1).

The function  $u_k$  is a differential polynomial  $u_k = \Theta_k(u_1, u'_1, \dots, u_1^{(2k)}, \mu_1, \dots, \mu_{k-1})$ . Let  $\varepsilon_k$  be a constant term of  $\Theta_k$ . Then  $\mathcal{U}_1 \mathcal{L}^{g-1} + \mathcal{U}_2 \mathcal{L}^{g-2} + \dots + \mathcal{U}_g = \partial_x^{2g+1} -$

$1/2 \sum u_k \partial_x^{2g-2k-1} + \sum \vartheta_i \partial_x^i = \partial_x^{2g+1} - 1/2 \sum \varepsilon_k \partial_x^{2g-2k-1} + \sum \tilde{\vartheta}_i \partial_x^i$  where  $\vartheta_i$  and  $\tilde{\vartheta}_i$  are differential polynomials in  $u_1$  without constant terms. On the other hand,  $A = \partial_x^{2g+1} - 1/2 \sum a_k \partial_x^{2k+1} + \sum \tilde{\vartheta}_i \partial_x^i$ . Thus,  $a_k = -1/2 \varepsilon_{g-k-1}$ , and so it remains to find  $\varepsilon_k$ . The result follows now from (4.1).  $\square$

The following corollary is one of the main results of the paper:

**Corollary 4.1.** *There is a canonical way to associate a solution  $u_1$  of the stationary  $g$ -KdV equation with a hyperelliptic curve*

$$(4.5) \quad \Gamma = \{(\xi, y) \in \mathbb{C}^2 \mid y^2 = 4\mu(\xi)\}.$$

The coefficients  $\mu_1, \dots, \mu_{g-1}$  are expressed in terms of the constants  $a_i$  as in equation 4.4, and  $\mu_g, \dots, \mu_{2g}$  are found from 4.3 in terms of the values of  $u_1^{(k)}(t_0)$ ,  $k = 0, 1, \dots$ , at some point  $t_0 \in \mathbb{C}^g$ .

*Remark 4.1.* The hyperelliptic curve constructed above coincides with the spectral curve introduced in [12] when the solution  $u_1$  is periodic as a function of  $x$ . Our construction uses only the local properties of the function  $u_1$ , while in [12] only periodical or rapidly decreasing functions are discussed.

*Remark 4.2.* The number of singular points on  $\Gamma$  is an important characteristic of the solution  $u_1$ . This number can be expressed in terms of  $u_1^{(k)}(t_0)$  using the resultant.

## 5. FIBER BUNDLES ASSOCIATED WITH THE STATIONARY $g$ -KdV EQUATIONS

The equations described by (1.9) are ordinary differential equations of order  $2g + 1$ , and so their solution  $u_1$  is *uniquely* determined in a neighbourhood of a given point  $x_0$  by the values  $c_k = u_1^{(k)}(x_0)$ ,  $k = 0, \dots, 2g$ . Since the coefficients of the KdV equations are constants, we can take  $x_0 = 0$ . The stationary  $g$ -KdV equations depend on the numbers  $a_0, \dots, a_{g-2}$  and so the space of all such equations is isomorphic to  $\mathbb{C}^{g-1}$ .

The space  $\mathcal{M}_g$  of all the hyperelliptic curves  $\Gamma = \{(\xi, y) \in \mathbb{C}^2 \mid y^2 = 4\mu(\xi)\}$  can be parametrized by the numbers  $\mu_1, \dots, \mu_{2g}$ , so it is isomorphic to  $\mathbb{C}^{2g}$ .

Denote by  $\mathbf{R}_g$  the space of solutions  $u$  of all the stationary  $g$ -KdV equations such that  $u$  is regular at the point  $x_0$ . As it was explained above, we can identify the space  $\mathbf{R}_g$  with  $\mathbb{C}^{3g}$  using coordinates  $(c_0, c_1, \dots, c_{2g}, a_0, \dots, a_{g-2})$ .

There exists a canonical map  $\pi_{\mathcal{M}} : \mathbf{R}_g \rightarrow \mathcal{M}_g$ , which sends a solution  $u$  to a hyperelliptic curve  $\Gamma$  described in (4.5).

Denote by  $\mathfrak{U}_g$  the space of  $g$ -th symmetric powers of hyperelliptic genus  $g$  curves. We consider the universal bundle  $(\mathfrak{U}_g, \mathcal{M}_g, \pi_{\mathfrak{U}})$  where the natural projection  $\pi_{\mathfrak{U}} : \mathfrak{U}_g \rightarrow \mathcal{M}_g$  is given by  $\pi_{\mathfrak{U}}(x \in \text{Sym}^g \Gamma) = \Gamma$ .

**Theorem 5.1.** *There exists a canonical fiber-preserving birational equivalence  $\mathbf{R}_g \rightarrow \mathcal{U}_g$ .*

**Proof.** Let  $\Gamma$  be a hyperelliptic curve associated with the solution  $u_1$  of the stationary  $g$ -KdV equation (see Corollary 4.1). Let  $\xi_1, \dots, \xi_g$  be the roots of the equation  $2 - \mathbf{u}(0, \xi) = 0$ . Denote  $y_i = \mathbf{u}'(0, \xi_i)$ . Equation (4.1) implies that  $y_i^2 = \mathbf{u}'(0, \xi_i)^2 = 4\mu(\xi_i)$ , so the point  $(\xi_i, y_i)$  belongs to  $\Gamma$ . Thus we have a map  $v : \mathbf{R}_g \rightarrow \mathfrak{U}_g$  given by the formula  $v(u_1) = (\Gamma, [(\xi_1, y_1), \dots, (\xi_g, y_g)])$  where  $(\xi_i, y_i) \in \Gamma$ . Apparently,  $v$  is fiber-preserving.

On the other hand, if a curve  $\Gamma$  and a point  $[(\xi_1, y_1), \dots, (\xi_g, y_g)] \in \text{Sym}^g \Gamma$  are given, then in the case of distinct points  $(\xi_1, \dots, \xi_g)$ , it is possible to construct the point  $(c_0, \dots, c_{2g}, a_0, \dots, a_{g-2})$  as follows. The constants  $a_k$  are a solution of (4.4) where the parameters  $\mu_i$  are known. The values  $u_i(0)$  are the symmetric functions of  $\xi_1, \dots, \xi_g$ , namely  $u_i(0) = 2(-1)^{g-i} \sigma_{g-i+1}(\xi_1, \dots, \xi_g) / \sigma_g(\xi_1, \dots, \xi_g)$ . Then the values  $u'_i(0)$  can be found, as the coefficients of the generating function  $\mathbf{u}'(0, \xi)$ , from the equations  $\mathbf{u}'(0, \xi_i) = y_i$ . All the higher derivatives  $c_k = u_1^{(k)}(0)$  can be found by recursion using equation (4.3). Thus the inverse rational map  $v^{-1}$  is constructed.  $\square$

In case of the universal bundle of Jacobians over the moduli space of genus  $g$  hyperelliptic curves this theorem gives the famous results of Dubrovin and Novikov, see [12].

## 6. ALGEBRAIC RELATIONS BETWEEN THE OPERATORS $\mathcal{L}, \mathcal{U}_1, \dots, \mathcal{U}_g$

The Burchall–Chaundy lemma ([7]) says that two commuting differential operators of one variable are always connected by an algebraic relation. In [16] the case of commuting differential operators of  $n$  variables was considered. In the same work they introduced a class of  $n$ -algebraic families of operators, i.e. families of commuting operators characterized by finite-dimensional algebraic manifolds. The family  $\{\mathcal{L}, \mathcal{U}_1, \dots, \mathcal{U}_g\}$  gives an example of  $n$ -algebraic operators from [16].

**Lemma 6.1.** *The operators  $\mathcal{L}, \mathcal{U}_1, \dots, \mathcal{U}_g$  satisfy the following algebraic relation:*

$$4(\mathcal{U}_1 \mathcal{L}^{g-1} + \mathcal{U}_2 \mathcal{L}^{g-2} + \dots + \mathcal{U}_{g-1} \mathcal{L} + \mathcal{U}_g)^2 = (4\mathcal{L}^{2g+1} + \mu_1 \mathcal{L}^{2g-1} + \mu_2 \mathcal{L}^{2g-2} + \dots + \mu_{2g}).$$

Using the notations  $\mathcal{U}(z) = \mathcal{U}_1 z^{g-1} + \mathcal{U}_2 z^g + \dots + \mathcal{U}_g$  and  $\tilde{\mu}(z) = 4z^{2g+1} + \mu_1 z^{2g-1} + \dots + \mu_{2g}$ , one can write down this relation as  $4\mathcal{U}(\mathcal{L})^2 = \tilde{\mu}(\mathcal{L})$ .

**Proof.** Denote

$$S_i = \frac{1}{2} u_i \partial_x - \frac{1}{4} u'_i.$$

Then

$$(6.1) \quad \mathcal{U}_i = \partial_i \mathcal{L} - S_i - \partial_{i+1}.$$

We have

$$[\mathcal{L}, S_i] = u'_i \mathcal{L} - u'_{i+1},$$

which implies the equation

$$\begin{aligned} \sum_{i+j=k} \mathcal{U}_i \mathcal{U}_j &= \sum_{i+j=k} (\partial_i \mathcal{L} - S_i - \partial_{i+1})(\partial_j \mathcal{L} - S_j - \partial_{j+1}) = \\ &= \sum_{i+j=k} \partial_i \partial_j \mathcal{L}^2 - 2\partial_i \partial_{j+1} \mathcal{L} + \partial_{i+1} \partial_{j+1} - (\partial_i S_j + S_i \partial_j) \mathcal{L} + (\partial_{i+1} S_j + S_i \partial_{j+1}) + S_i S_j. \end{aligned}$$

A direct calculation gives that

$$\begin{aligned} S_i S_j &= 1/4 u_i u_j \partial_x^2 + 1/8 (u_i u'_j - u_j u'_i) \partial_x - 1/8 u_i u''_j + 1/16 u'_i u'_j, \\ \partial_x S_i + S_i \partial_x &= u_i \partial_x^2 - 1/4 u''_i. \end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{1 \leq i, j \leq g} \mathcal{U}_i \mathcal{U}_j \mathcal{L}^{2g-i-j+1} &= \partial_x^2 \mathcal{L}^{2g} - \sum_{1 \leq i \leq g} (u_i \partial_x^2 - 1/4 u_i'') \mathcal{L}^{2g-i} + \\
&+ \sum_{1 \leq i, j \leq g} (1/4 u_i u_j \partial_1^2 - 1/2 u_i u_j'' + 1/16 u_i' u_j') \mathcal{L}^{2g-i-j} = \\
&\mathcal{L}^{2g+1} + u_1 \mathcal{L}^{2g} - \sum_{1 \leq i \leq g} u_i \mathcal{L}^{2g-i+1} - \sum_{1 \leq i \leq g} (u_i u_1 - 1/4 u_i'') \mathcal{L}^{2g-i-1} + \\
&+ \sum_{1 \leq i, j \leq g} (1/4 u_i u_j \mathcal{L}^{2g-i-j+1}) + \sum_{1 \leq i, j \leq g} (1/4 u_i u_j u_1 - 1/8 u_i u_j'' + 1/16 u_i' u_j') \mathcal{L}^{2g-i-j}.
\end{aligned}$$

We see that the coefficient at  $\mathcal{L}^{2g-i}$  in this formula is exactly the coefficient at  $\xi^i$  in the expression  $1/16(\mathbf{u}'(\xi)^2 + 2\mathbf{u}''(\xi)(2 - \mathbf{u}(\xi)) + 4(u_1 + \xi^{-1})(2 - \mathbf{u}(\xi))^2 = 1/4\mu(\xi)$ .  $\square$

**Corollary 6.1.** *Let  $\Psi(t_1, t_2, \dots, t_g)$  be a common eigenfunction of the operators  $\mathcal{L}, \mathcal{U}_1, \dots, \mathcal{U}_g$ , with the eigenvalues  $E, \alpha_1, \dots, \alpha_g$ . Let  $\xi = E^{-1}$  and  $\alpha(\xi) = \sum_{i=1}^g \alpha_i \xi^i$ . Then*

$$(6.2) \quad 4\alpha(\xi)^2 = \mu(\xi).$$

## 7. A COMMON EIGENFUNCTION OF THE FAMILY $\{\mathcal{U}_i\}$

In this section we construct a common eigenfunction of the family of commuting differential operators  $\{\mathcal{U}_i\}$ .

**Lemma 7.1.**

$$\frac{\partial}{\partial t_i} \left( \frac{D_j(2 - \mathbf{u}(\xi))}{2 - \mathbf{u}(\xi)} \right) = \frac{\partial}{\partial t_j} \left( \frac{D_i(2 - \mathbf{u}(\xi))}{2 - \mathbf{u}(\xi)} \right)$$

**Proof.** It follows from the definition of the operators  $\partial(\eta)$  and  $D_i$  that the expression  $\frac{\partial}{\partial t_i} \left( \frac{D_j(2 - \mathbf{u}(\xi))}{2 - \mathbf{u}(\xi)} \right)$  equals to the coefficient at  $\zeta^i \eta^j$  in the expansion  $\partial(\zeta) \frac{\xi \eta}{2(\xi - \eta)} \frac{2 - \mathbf{u}(\eta)}{2 - \mathbf{u}(\xi)}$  with respect to  $\eta$  and  $\zeta$ . This function is equal to  $\frac{1}{(2 - \mathbf{u}(\xi))^2} \mathcal{T}_\xi^\eta \mathcal{T}_\xi^\zeta \mathbf{u}'(\xi)$  (see (3.10)). Since the generalized translation  $\mathcal{T}_\xi^\eta$  is commutative, this function is symmetric with respect to the variables  $\zeta$  and  $\eta$ . Consequently the coefficients of  $\zeta^i \eta^j$  and  $\zeta^j \eta^i$  are equal.  $\square$

**Corollary 7.1.** *There exists a function  $F(\xi) = F(t_1, \dots, t_g, \xi)$  such that  $\partial_i F = \frac{D_i(2 - \mathbf{u}(\xi))}{2 - \mathbf{u}(\xi)}$ ,  $1 \leq i \leq g$ . The function  $F(\xi)$  is uniquely defined up to an additive constant in a neighborhood of any point  $(\bar{t}_0, \xi_0) = (t_1^0, \dots, t_g^0, \xi_0)$  such that  $2 - \mathbf{u}(\bar{t}_0; \xi_0) \neq 0$ .*

Consider also the function  $\Phi = \Phi(t_1, \dots, t_g; E = \xi^{-1}, \alpha_1, \dots, \alpha_g)$  given by

$$(7.1) \quad \Phi = \sqrt{2 - \mathbf{u}(\xi)} \exp(2\alpha(\xi)F(\xi)) \exp(-2 \sum_{i=1}^g D_i(\alpha(\xi))t_i),$$

where  $\alpha(\xi) = \sum_{i=1}^g \alpha_i \xi^i$ . The function  $\Phi$  is uniquely defined up to a multiplicative constant in a neighborhood of any point  $(\bar{t}_0, \xi_0, \bar{\alpha})$  such that  $2 - \mathbf{u}(\bar{t}_0; \xi_0) \neq 0$ .

Find the derivatives of the function  $\Phi$  with respect to  $x = t_1$  and  $t_k$ ,  $k \geq 2$ :

$$(7.2) \quad \Phi' = \frac{4\alpha(\xi) - \mathbf{u}'(\xi)}{2(2 - \mathbf{u}(\xi))} \Phi,$$

$$(7.3) \quad \begin{aligned} \Phi'' &= \left( \frac{-\mathbf{u}''(\xi)(\mathbf{u}(\xi) + \mathbf{u}'(\xi)(4\alpha(\xi) - \mathbf{u}'(\xi)))}{2(2 - \mathbf{u}(\xi))} + \frac{(4\alpha(\xi) - \mathbf{u}'(\xi))^2}{4(2 - \mathbf{u}(\xi))^2} \right) \Phi \\ &= \frac{16\alpha(\xi)^2 - 2\mathbf{u}''(\xi)(2 - \mathbf{u}(\xi)) - \mathbf{u}'(\xi)^2}{4(2 - \mathbf{u}(\xi))^2} \Phi, \end{aligned}$$

$$(7.4) \quad \partial_k \Phi = \left( \frac{4\alpha(\xi)D_k(2 - \mathbf{u}(\xi)) - \partial_k \mathbf{u}(\xi)}{2(2 - \mathbf{u}(\xi))} - 2D_k(\alpha(\xi)) \right) \Phi.$$

**Lemma 7.2.** *The function  $\Phi$  is an eigenfunction of the operator  $\mathcal{L}$  with the eigenvalue  $E \neq 0$  if and only if  $\xi = E^{-1}$  and  $\{\xi, 4\alpha(\xi)\} \in \Gamma$ , where  $\Gamma$  is a curve defined by equation (4.5).*

**Proof.** Equation (4.1) implies that

$$(\mathcal{L} - E)\Phi = \left( \frac{4\alpha(\xi)^2 - \mu(\xi)}{2 - \mathbf{u}(\xi)} - (\xi^{-1} - E) \right) \Phi.$$

The function in parentheses vanishes identically iff  $4\alpha(\xi)^2 - \mu(\xi) = (\xi^{-1} - E)(2 - \mathbf{u}(\xi))$ . Differentiating the last formula with respect to  $x$ , we obtain that  $(\xi^{-1} - E)\mathbf{u}'(\xi) = 0$ . Hence  $\xi^{-1} = E$  and  $4\alpha(\xi)^2 = \mu(\xi)$ .  $\square$

**Theorem 7.1.** *Suppose that  $4\alpha(\xi)^2 = \mu(\xi)$ . Then*

- (1) *The function  $\Phi$  is a common eigenfunction of the family  $\mathcal{L}, \mathcal{U}_1, \dots, \mathcal{U}_g$  with eigenvalues  $E = \xi^{-1}, \alpha_1, \dots, \alpha_g$ .*
- (2) *The space of common eigenfunction of operators  $\mathcal{L}, \mathcal{U}_1, \dots, \mathcal{U}_g$  with eigenvalues  $E = \xi^{-1}, \alpha_1, \dots, \alpha_g$  is one-dimensional.*

**Proof.** Express the operators  $\mathcal{U}_k$  as

$$(7.5) \quad \mathcal{U}_k = \partial_k(\partial_x^2 - (u_1 + \xi^{-1})) + \xi^{-1}\partial_k - 1/2u_k\partial_x + 1/4u'_k - \partial_{k+1}.$$

Let  $\Psi$  be a common eigenfunction of  $\mathcal{L}, \mathcal{U}_k$  with the eigenvalues mentioned above. Then

$$(7.6) \quad (\xi^{-1}\partial_k - 1/2u_k\partial_x + 1/4u'_k - \partial_{k+1})\Psi = \alpha_k\Psi.$$

This allows to express all the partial derivatives  $\partial_k\Psi$  in terms of  $\Psi$  and  $\Psi'$ , namely

$$\partial_k\Psi = D_k(2 - \mathbf{u}(\xi))\Psi' + \frac{1}{2}D_k(\mathbf{u}'(\xi))\Psi - 2D_k(\alpha(\xi))\Psi, \quad 1 \leq k \leq g-1.$$

For  $k = g-1$  one gets from (7.6)

$$\xi^{-1}\partial_{g-1}\Psi - 1/2u_{g-1}\Psi' + 1/4u'_{g-1}\Psi = \alpha_g\Psi.$$

Therefore,

$$\xi^{-1} \left( D_g(2 - \mathbf{u}(\xi))\Psi' + \frac{1}{2}D_g(\mathbf{u}'(\xi))\Psi - 2D_g(\alpha(\xi))\Psi \right) - 1/2u_g\Psi' + 1/4u'_g\Psi - \alpha_g\Psi = 0.$$

Using Lemma 3.4 and (3.2) we obtain

$$(2 - \mathbf{u}(\xi))\Psi' = (1/2\mathbf{u}'(\xi) + \alpha(\xi))\Psi.$$

Thus,

$$\frac{\Psi'}{\Psi} = \frac{-1/2\mathbf{u}'(\xi) + \alpha(\xi)}{2 - \mathbf{u}(\xi)},$$

and

$$\begin{aligned} \frac{\partial_k \Psi}{\Psi} &= D_k(2 - \mathbf{u}(\xi)) \frac{-1/2 \mathbf{u}'(\xi) + \alpha(\xi)}{2 - \mathbf{u}(\xi)} + \frac{1}{2} D_k(\mathbf{u}'(\xi)) - 2D_k(\alpha(\xi)) \\ &= \frac{\alpha(\xi) D_k(2 - \mathbf{u}(\xi)) - 1/2 \partial_k \mathbf{u}(\xi)}{2 - \mathbf{u}(\xi)} - 2D_k(\alpha(\xi)), \quad k = 2, \dots, g. \end{aligned}$$

We see that  $\frac{\partial_k \Psi}{\Psi} = \frac{\partial_k \Phi}{\Phi}$ ,  $k = 1, \dots, g$ . Therefore,  $\Psi = \lambda \Phi$  where  $\lambda$  is a constant.  $\square$

Consider now the special case  $E = 0$ .

**Theorem 7.2.** *The space of common eigenfunction of operators  $\mathcal{L}, \mathcal{U}_1, \dots, \mathcal{U}_g$  with eigenvalues  $0, \alpha_1, \dots, \alpha_g$ , where  $4\alpha_g^2 = \mu_{2g}$ , is one-dimensional.*

**Proof.** Let  $\Phi^0$  be a common eigenfunction of operators  $\mathcal{L}, \mathcal{U}_1, \dots, \mathcal{U}_g$  with eigenvalues  $0, \alpha_1, \dots, \alpha_g$ . Since  $\mathcal{L}\Phi^0 = 0$ , equation (6.1) implies that

$$\mathcal{U}_k \Phi^0 = (-1/2 u_k \partial_x + 1/4 u'_k - \partial_{k+1}) \Phi^0 = \alpha_k \Phi^0.$$

Therefore,

$$(7.7) \quad \partial_k \Phi^0 = -(1/2 u_{k-1} \partial_x - 1/4 u'_{k-1} + \alpha_{k-1}) \Phi^0, \quad k = 2, \dots, g$$

and

$$(7.8) \quad \partial_x \Phi^0 = \frac{u'_g - 4\alpha_g}{2u_g} \Phi^0.$$

Note that (7.8) and (4.1) imply

$$\partial_x^2 \Phi^0 = \frac{2u''_g u_g - (u'_g)^2 + 16\alpha_g^2}{4u_g^2} \Phi^0 = \frac{2u''_g u_g - (u'_g)^2 + 4\mu_{2g}}{4u_g^2} \Phi^0 = u \Phi^0.$$

It follows from (7.7) and (7.8) that

$$\partial_k \Phi^0 = \frac{u_{k-1}(4\alpha_g - u'_g) - u_g(4\alpha_{k-1} - u'_{k-1})}{4u_g} \Phi^0.$$

Since the logarithmic derivatives are uniquely defined, the space of eigenfunction with eigenvalues  $E = 0, \alpha_1, \dots, \alpha_g = 1/4\sqrt{\mu_{2g}}$  is one-dimensional.  $\square$

Note that the function  $\Phi$  can be expressed as

$$(7.9) \quad \Phi = \exp \tilde{F},$$

where

$$(7.10) \quad \tilde{F} = \left( 2\alpha(\xi)F - 2 \sum_{i=1}^g D_i(\alpha(\xi))t_i + \frac{1}{2} \log(2 - \mathbf{u}(\xi)) \right).$$

We have

$$(7.11) \quad \partial_i \tilde{F} = \frac{D_i(2 - \mathbf{u}(\xi))(4\alpha(\xi) - \mathbf{u}'(\xi)) - D_i(4\alpha(\xi) - \mathbf{u}'(\xi))(2 - \mathbf{u}(\xi))}{2(2 - \mathbf{u}(\xi))}.$$

Using the notation  $\partial(\eta) = \sum_{i=1}^g \eta^i \partial_i$ , we can rewrite these formulas as

$$\partial(\eta) \tilde{F} = \frac{\xi \eta}{\xi - \eta} \frac{(2 - \mathbf{u}(\eta))(4\alpha(\xi) - \mathbf{u}'(\xi)) - (4\alpha(\eta) - \mathbf{u}'(\eta))(2 - \mathbf{u}(\xi))}{4(2 - \mathbf{u}(\xi))} = \frac{T_\xi^\eta(4\alpha(\xi) - \mathbf{u}'(\xi))}{2(2 - \mathbf{u}(\xi))}.$$



**Lemma 7.3.** *The function  $\chi(\xi) = \partial_1 \tilde{F} = \frac{4\alpha(\xi) - \mathbf{u}'(\xi)}{2(2 - \mathbf{u}(\xi))}$  satisfies the Riccati equation  $\chi'(\xi) + \chi(\xi)^2 = u_1 + \xi^{-1}$ . Moreover,*

$$\frac{1}{2 - \mathbf{u}(\eta)} \partial(\eta) \tilde{F} = \frac{\xi \eta}{2(\xi - \eta)} (\chi(\xi) - \chi(\eta)).$$

Denote by  $V$  a hypersurface in  $\mathbb{C}^{g+1} = \{(\xi, \alpha_1, \dots, \alpha_g)\}$  defined by equation (6.2). Recall that  $\Gamma = \{\xi, y \in \mathbb{C}^2 \mid y^2 = 4\mu(\xi)\}$ . In coordinates  $E, \alpha_i$  the hypersurface  $V$  is given by the equation

$$4(\alpha_1 E^{g-1} + \alpha_2 E^{g-2} + \dots + \alpha_g)^2 = 4E^{2g+1} + \mu_1 E^{2g-1} + \dots + \mu_{2g},$$

and the curve  $\Gamma$  is given by the equation  $\eta^2 = 4(4E^{2g+1} + \mu_1 E^{2g-1} + \dots + \mu_{2g})$  where  $\eta = y\xi^{-g}$ .

Define a projection  $\pi : V \rightarrow \Gamma$  by the formula  $\pi(\xi, \alpha_1, \dots, \alpha_g) = (\xi, 2\alpha(\xi))$ . In the sequel we will consider the curve  $\Gamma$  as subvariety of  $V$  using a canonical embedding  $i : \Gamma \hookrightarrow V$  defined as  $i(\xi, \eta) = (\xi, 0, 0, \dots, \eta)$ . Let  $V^* = \pi^{-1}(\Gamma^*)$  where  $\Gamma^* = \{(\xi, y) \in \Gamma; \xi \neq 0\}$ .

Recall that  $\mathbb{C}^* = \{\xi \in \mathbb{C} \mid \xi \neq 0\}$ . The function  $\Phi$  of equation (7.1) is defined in the space  $\mathbb{C}^g \times \mathbb{C}^* \times \mathbb{C}^g$  parametrized with coordinates  $t_1, \dots, t_g, \xi, \alpha_1, \dots, \alpha_g$ . Consider this domain as a graded space using the following grading:  $\deg t_k = 1 - 2k$ ,  $\deg \xi = -2$ ,  $\deg \alpha_k = 2k + 1$ . Take also  $\deg \mu_i = 2i + 2$ . Then the equation  $4\alpha(\xi) = \mu(\xi)$  defining the variety  $V^*$  is homogeneous.

**Lemma 7.4.** *Let  $\Phi$  be a common eigenfunction of the operators  $\mathcal{L}, \mathcal{U}_1, \dots, \mathcal{U}_g$  with eigenvalues  $E = \xi^{-1}, \alpha_1, \dots, \alpha_g$ . Let  $\gamma_2, \dots, \gamma_g \in \mathbb{C}$  be arbitrary constants. Then the function  $\tilde{\Phi} = \Phi \exp(\gamma_2 t_2 + \dots + \gamma_g t_g)$  is also a common eigenfunction of these operators, its eigenvalues given by  $E = \xi^{-1}$ ,  $\tilde{\alpha}_1 = \alpha_1 - \gamma_2$ ,  $\tilde{\alpha}_2 = \alpha_2 - \gamma_3 + \xi \gamma_2, \dots, \tilde{\alpha}_i = \alpha_i - \gamma_{i+1} + \xi \gamma_i, \dots, \tilde{\alpha}_g = \alpha_g + \xi \gamma_g$ .*

**Proof.** Take  $\gamma_1 = 0$ . It is obvious that  $\frac{\partial_k \tilde{\Phi}}{\tilde{\Phi}} = \frac{\partial_k \Phi}{\Phi} + \gamma_k$ ,  $k = 1, \dots, g$  and  $\mathcal{L}\tilde{\Phi} = E\tilde{\Phi}$ . From (7.5) one obtains that

$$\mathcal{U}_k \tilde{\Phi} = (\xi^{-1} \partial_k - \frac{1}{2} u_k \partial_x + \frac{1}{4} u'_k - \partial_{k+1}) \tilde{\Phi} = \exp(\gamma_2 t_2 + \dots + \gamma_g t_g) (\mathcal{U}_k \Phi + (\xi^{-1} \gamma_k - \gamma_{k+1}) \Phi).$$

Therefore,

$$\frac{\mathcal{U}_k \tilde{\Phi}}{\tilde{\Phi}} = \tilde{\alpha}_k,$$

where

$$(7.12) \quad \tilde{\alpha}_k = \alpha_k + (\xi^{-1} \gamma_k - \gamma_{k+1}). \quad \square$$

Note that  $\sum_{i=1}^g \tilde{\alpha}_i \xi^i = \sum_{i=1}^g \alpha_i \xi^i$ .

Assume that  $\deg \gamma_k = 2k - 1$ .

**Corollary 7.2.** *Equation (7.12) defines a free action of the graded additive group  $\mathbb{C}^{g-1}$  with coordinates  $\gamma_2, \dots, \gamma_g$  on the variety  $V^*$ . The quotient space  $V^*/\mathbb{C}^{g-1}$  is  $\Gamma^*$ . The vector bundle  $V^* \rightarrow \Gamma^*$  is trivial.*

**Proof.** Define the map  $s : \Gamma^* \times \mathbb{C}^{g-1} \rightarrow V^*$  by the formula  $s(\xi, y, \gamma_2, \dots, \gamma_{g-2}) = (\xi, t, -\gamma_2, \xi^{-1} \gamma_2 - \gamma_3, \dots, \xi^{-1} \gamma_g)$ . This is the required trivialization.  $\square$

Consider the case  $u_1 \equiv 0$ . In this case the operators  $\mathcal{U}_k$  and  $\mathcal{L}$  are

$$(7.13) \quad \mathcal{L} = \partial_x^2, \quad \mathcal{U}_k = \partial_x^2 \partial_k - \partial_{k+1}.$$

**Lemma 7.5.** *Let  $\alpha_1, \dots, \alpha_g$  and  $\xi$  satisfy the equation  $(\sum_{i=1}^g \alpha_i \xi^i)^2 = \xi^{-1}$ . Then the function*

$$(7.14) \quad \Phi_0 = \exp \left( \sum_{1 \leq k \leq i \leq g} \alpha_i t_k \xi^{i-k+1} \right)$$

*is a common eigenfunction of the operators (7.13) with eigenvalues  $E = \xi^{-1}, \alpha_1, \dots, \alpha_g$ .*

**Proof.** The logarithmic derivative of the function  $\Phi_0$  are given by

$$\frac{\partial_k \Phi_0}{\Phi_0} = \sum_{i=k}^g \alpha_i \xi^{i-k+1}$$

It is clear that  $\partial_x^2 \Phi_0 = \xi^{-1} \Phi_0$  and  $\mathcal{U}_k \Phi_0 = (\xi^{-1} \partial_k - \partial_{k+1}) \Phi_0 = \alpha_k \Phi_0$ .  $\square$

The function  $\Phi_0$  can be obtained from the formula (7.1) by rescaling. This fact will be proved in Section 10.2.

## 8. BASIC GENERATING FUNCTION FOR THE SOLUTION OF STATIONARY $g$ -KDV EQUATION

Denote  $\mu(\xi, \eta) = 4\xi^{-1} + 4\eta^{-1} + 2 \sum_{i=1}^g \mu_{2i} \xi^i \eta^i + \sum_{i=0}^{g-1} \mu_{2i+1} (\xi + \eta) \xi^i \eta^i$ . We have  $\mu(\xi, \xi) = 2\mu(\xi)$  and  $\mu(\xi, \eta) = \mu(\eta, \xi)$ , so  $\mu(\xi, \eta)$  is a polarization of  $\mu(\xi)$  (see Definition 3.1).

Consider the function

$$Q(\xi, \eta) = \mathbf{u}'(\xi) \mathbf{u}'(\eta) + (2 - \mathbf{u}(\xi)) \mathbf{u}''(\eta) + \mathbf{u}''(\xi) (2 - \mathbf{u}(\eta)) + 2(2 - \mathbf{u}(\xi))(2 - \mathbf{u}(\eta)) (\xi^{-1} + \eta^{-1} + 2u_1).$$

The function  $Q(\xi, \eta)$  is a polarization of the function in the right-hand side of (4.1). Therefore  $Q(\xi, \xi) = \mu(\xi, \xi)$ . Equations (3.1) and (4.1) imply that

$$\left. \frac{\partial \mu(\xi, \eta)}{\partial \xi} \right|_{\xi=\eta} = \left. \frac{\partial Q(\xi, \eta)}{\partial \xi} \right|_{\xi=\eta}.$$

Denote also

$$P(\xi) = \frac{\xi^4}{8} \left( \left. \frac{\partial^2 \mu(\xi, \eta)}{\partial \xi \partial \eta} \right|_{\xi=\eta} - \left. \frac{\partial^2 Q(\xi, \eta)}{\partial \xi \partial \eta} \right|_{\xi=\eta} \right).$$

**Lemma 8.1.** *The function*

$$(8.1) \quad P(\xi, \eta) = \frac{\xi^2 \eta^2}{4(\xi - \eta)^2} (2\mu(\xi, \eta) - Q(\xi, \eta)).$$

*is a polarization of  $P(\xi)$ .*

**Proof.** It is obvious that  $P(\xi, \eta)$  is symmetric. Direct calculations show that

$$\begin{aligned} P(\xi) = \frac{\xi^4}{8} & \left( 2 \left. \frac{\partial^2 \mu(\xi, \eta)}{\partial \xi^2} \right|_{\xi=\eta} - 2 \frac{\partial^2 \mu(\xi)}{\partial \xi^2} + \left( \frac{\partial \mathbf{u}'(\xi)}{\partial \xi} \right)^2 - 2 \frac{\partial \mathbf{u}''(\xi)}{\partial \xi} \frac{\partial \mathbf{u}(\xi)}{\partial \xi} \right. \\ & \left. + 4 \xi^{-2} \frac{\partial \mathbf{u}(\xi)}{\partial \xi} (2 - \mathbf{u}(\xi)) + 4(\xi^{-1} + u_1) \left( \frac{\partial \mathbf{u}(\xi)}{\partial \xi} \right)^2 \right) = \frac{1}{2} \lim_{\xi \rightarrow \eta} P(\xi, \eta). \quad \square \end{aligned}$$

**Corollary 8.1.**  *$P(\xi, \eta)$  is a polynomial of degree  $g$  in variables  $\xi$  and  $\eta$ ,*

Define functions  $p_{ij}$  as coefficients in the expansion

$$(8.2) \quad P(\xi, \eta) = \sum_{i=1}^g \sum_{j=1}^g p_{ij} \xi^i \eta^j.$$

**Lemma 8.2.**  $p_{1i} = p_{i1} = u_i$

**Proof.** Lemma follows from the formula  $\sum_{i=1}^g p_{1i} \xi^i = \frac{P(\xi, \eta)}{\eta} \Big|_{\eta \rightarrow 0} = \mathbf{u}(\xi)$ .  $\square$

This result motivates the following definition:

**Definition 8.1.** The function  $P(\xi, \eta)$  is called the basic generating function for the solution  $u_1$  of the stationary KdV equation.

The coefficient at  $\eta^2$  in (8.1) is equal to :

$$(8.3) \quad 2 \sum_{i=1}^g p_{2i} \xi^i = -3(2 - \mathbf{u}(\xi))(u_1 + 2\xi^{-1}) - \mathbf{u}''(\xi) + \mu_1 \xi + 12\xi^{-1}.$$

Therefore,

$$(8.4) \quad u_i'' = 3u_i u_1 + 6u_{i+1} - 2p_{2,i} + \mu_1 \delta_{1i}$$

where  $\delta_{ij}$  is the Kronecker symbol.

It will be shown later (see section 10.1) that if  $u_1 = 2\wp_{gg}$  is a solution of the stationary KdV equation from [3], then  $u_i = 2\wp_{g,g-i+1}$ ,  $u_i'' = 2\wp_{ggg,g-i+1}$ ,  $p_{2,i} = 2\wp_{g-1,g-i+1}$ . equation (8.4) becomes the basic relation for  $\wp$ -functions (see (4.1) in [3]). All the results of [3] for the  $\wp$ -functions, derived from the basic relation, are thus true for the arbitrary solution of the stationary KdV.

**Lemma 8.3.**  $\partial(\zeta)P(\xi, \eta) = \partial(\xi)P(\zeta, \eta)$ .

**Proof.** We have

$$\begin{aligned} \partial(\zeta)P(\xi, \eta) &= \frac{\xi^2 \eta^2 \zeta^2}{8(\xi - \eta)(\xi - \zeta)(\eta - \zeta)} \left( \mathbf{u}''(\xi)(\mathbf{u}'(\eta)(2 - \mathbf{u}(\zeta)) - \mathbf{u}'(\zeta)(2 - \mathbf{u}(\eta))) \right. \\ &\quad \left. - \mathbf{u}''(\eta)(\mathbf{u}'(\xi)(2 - \mathbf{u}(\zeta)) - \mathbf{u}'(\zeta)(2 - \mathbf{u}(\xi))) + \mathbf{u}''(\zeta)(\mathbf{u}'(\xi)(2 - \mathbf{u}(\eta)) - \mathbf{u}'(\eta)(2 - \mathbf{u}(\xi))) \right) \\ &\quad + \frac{\xi \eta \zeta^2}{4(\xi - \zeta)(\eta - \zeta)} \mathbf{u}'(\zeta)(2 - \mathbf{u}(\xi))(2 - \mathbf{u}(\eta)) - \frac{\xi \eta^2 \zeta}{4(\xi - \eta)(\eta - \zeta)} \mathbf{u}'(\eta)(2 - \mathbf{u}(\xi))(2 - \mathbf{u}(\zeta)) \\ &\quad + \frac{\xi^2 \eta \zeta}{4(\xi - \eta)(\xi - \zeta)} \mathbf{u}'(\xi)(2 - \mathbf{u}(\eta))(2 - \mathbf{u}(\zeta)) \\ &= \frac{1}{2} \mathcal{B}_3(\mathbf{u}''(\xi), \mathbf{u}'(\xi), (2 - \mathbf{u}(\xi)) + \mathcal{B}_3(\mathbf{u}'(\xi), 2 - \mathbf{u}(\xi), (2 - \mathbf{u}(\xi))\xi^{-1}) \end{aligned}$$

Thus,  $\partial(\zeta)P(\xi, \eta)$  is symmetric as a function of variables  $\xi, \eta, \zeta$ .  $\square$

**Corollary 8.2.** There exists a function  $\phi = \phi(t_1, \dots, t_g)$  such that  $P(\xi, \eta) = \partial(\xi)\partial(\eta)\phi$ .

**Corollary 8.3.**  $P'(\xi, \eta) = \partial(\eta)\mathbf{u}(\xi)$ .

**Proof.** Indeed,

$$P'(\xi, \eta) = \frac{\partial(\zeta)P(\xi, \eta)}{\zeta} \Big|_{\zeta \rightarrow 0} = \frac{\xi \eta}{2(\xi - \eta)} (\mathbf{u}'(\xi)(2 - \mathbf{u}(\eta)) - (2 - \mathbf{u}(\xi))\mathbf{u}'(\eta)) = \partial(\eta)\mathbf{u}(\xi). \quad \square$$

Note that it follows from Theorem 3.2 that  $\partial_x P(\xi, \eta) = \mathcal{T}_\xi^\eta \partial_x \mathbf{u}(\xi)$ .

9. A CONSTRUCTION OF THE  $w$ -FUNCTION

Consider the equation

$$(9.1) \quad 2\partial_x^2 \log w = -u_1,$$

with the initial conditions

$$(9.2) \quad w(0) = 1, \quad \partial_k w(0) = 0, \quad k = 1, \dots, g,$$

here  $u_1 = u_1(t_1, \dots, t_g)$  is a solution of the stationary  $g$ -KdV equation with respect to  $x = t_1$ .

**Theorem 9.1.** *There exists a differentiable solution  $w$  of (9.1), (9.2) such that the functions*

$$(9.3) \quad u_k = -2\partial_x \partial_k \log w, \quad k = 1, \dots, g$$

*satisfy the hypotheses of Theorem 4.2.*

**Definition 9.1.** *The solutions of (9.1) described in Theorem 9.1 are called special.*

**Theorem 9.2.** *Let  $p_{ij}(t)$  be as in (8.1) and (8.2). Then there is a unique special solution of (9.1) such that*

$$(9.4) \quad 2\partial_i \partial_j \log w = -p_{ij}$$

*for all  $i, j$ .*

**Proof.** The existence of a required solution of (9.4) follows from Corollary 8.2. The function  $w$  is defined by (9.4) up to a factor  $\exp(\lambda_0 + \lambda_1 t_1 + \dots + \lambda_g t_g)$ . All the constants  $\lambda_i$  are uniquely determined by the initial conditions (9.2).  $\square$

This result completed the solution of Problem 1.

**Definition 9.2.** *The special solution (9.1) described in Theorem 9.2 is called a  $w$ -function of the solution  $u$  of the stationary  $g$ -KdV equation.*

The relations between the higher logarithmic derivatives of the  $w$  function are obtained with the technique of generating function. For example  $\sum_{ijk} \xi^i \eta^j \zeta^k \partial_i \partial_j \partial_k \log w = \partial(\zeta)P(\xi, \eta)$ . This function was calculated in Lemma 8.3.

The solution  $u$  is a point of the space  $\mathbf{R}_g$  (see Section 5). Consequently, we can consider the  $w$ -function as a function  $w : \mathbb{C}^g \times \mathbf{R}_g \rightarrow \mathbb{C}$ .

The rest of this section is devoted to an explicit construction of the  $w$ -function starting with the given solution  $u$  of the KdV equation.

Denote  $t = (x, t_2, \dots, t_g)$  and put

$$\varphi(t) = \frac{1}{2} \int_0^x \int_0^x u(t) dx$$

Then equation (9.1) implies:  $w(t) = \exp(a(t) - \varphi(t))$  where  $a''(t) = 0$ . Therefore,  $a(t) = a_1(\tilde{t})x + a_0(\tilde{t})$  where  $\tilde{t} = (t_2, \dots, t_g)$ . The initial condition (9.2) gives now  $a_0(0) = 0$ ,  $a_1(0) = 0$ , and  $\partial_k a_0(0) = 0$ ,  $k = 2, \dots, g$ . It follows from (9.3) that

$$(9.5) \quad 2\partial_k a_1(\tilde{t}) = -u_k + 2\partial_k \int_0^x u(t) dt, \quad k = 2, \dots, g.$$

The set of equations (9.5) with the initial condition  $a_1(0) = 0$  has a unique solution  $a_1(\tilde{t})$ . It follows from (9.4) that  $2\partial_i \partial_j a_0(\tilde{t}) = 2\partial_i \partial_j \varphi(t) - 2\partial_i \partial_j a_1(\tilde{t})x - p_{ij}(t)$ . These equations with the initial condition  $a_0(0) = 0$ ,  $\partial_k a_0(0) = 0$ ,  $k = 1, \dots, g$  have a unique solution  $a_0(\tilde{t})$ .

## 10. APPLICATIONS

**10.1. Kleinian  $\sigma$ -function.** Consider hyperelliptic Kleinian functions  $\sigma(t)$ ,  $\zeta_i(t) = \partial_i \log \sigma(t)$ , and  $\wp_{ij}(t) = -2\partial_i \partial_j \log \sigma(t)$ . The function  $2\wp_{gg}(t)$  is a solution of the stationary KdV equation (see [3]).

**Corollary 10.1.** *Let  $z \in \mathbb{C}^g$  be a point where  $\sigma(z) \neq 0$ . Then the function*

$$w(t) = \frac{\sigma(t+z)}{\sigma(z)} \exp \langle -\zeta(z), t \rangle$$

*is a  $w$ -function of the solution  $2\wp_{gg}(t+z)$ .*

**Proof.** The functions  $u_i = 2\wp_{g,g-i+1}$  and  $p_{ij} = 2\wp_{g-i+1,g-j+1}$  satisfy equations (8.1), (9.4) (see [2]). The corollary now follows from the uniqueness of the  $w$ -function.  $\square$

Let  $\theta_g$  be the polynomials from [1]. The second logarithmic derivatives of  $\theta_g$  give solutions of the higher KdV equations. As it was proved in [4], the polynomial  $\theta_g$  is, up to a linear change of variables, a rational limit  $\widehat{\sigma}_g$  of the  $\sigma$ -function of genus  $g$ . Denote  $\widehat{\zeta}_i(t) = \partial_i \log \widehat{\sigma}_g(t)$ .

**Corollary 10.2.** *Let  $z \in \mathbb{C}^g$  be a point where  $\widehat{\sigma}_g(z) \neq 0$ . Then the function*

$$w(t) = \frac{\widehat{\sigma}_g(t+z)}{\widehat{\sigma}_g(z)} \exp \langle -\widehat{\zeta}(z), t \rangle$$

*is the  $w$ -function of the solution  $u = -2(\log \theta_g)''$ .*

**10.2. The homogeneity condition.** The results obtained in this section follow from the uniqueness theorems for the  $w$ -functions.

**Lemma 10.1.** *Suppose that  $u(x, t_2, \dots, t_g)$  is a solution of a stationary  $g$ -KdV equation with respect to  $x$ . Take  $\kappa \in \mathbb{C}^*$ . Then the function  $\widehat{u}(x, t_2, \dots, t_g) = \kappa^2 u(\kappa x, \kappa^3 t_2, \dots, \kappa^{2g-1} t_g)$  is also a solution of the stationary  $g$ -KdV equation. Under the transformation  $u \rightarrow \widehat{u}$  the constants  $\mu_i$  and  $a_i$  are transformed as  $\widehat{\mu}_i = \mu_i \kappa^{2i+2}$ ,  $\widehat{a}_i = \kappa^{2g-2i} a_i$ .*

**Proof.** Let  $\{u_1 = u, u_2, \dots, u_g\}$  be a sequence of functions from Theorem 4.2. Then the functions

$$(10.1) \quad \widehat{u}_i(x, t_2, \dots, t_g) = \kappa^{2i} u_i(\kappa x, \kappa^3 t_2, \dots, \kappa^{2g-1} t_g)$$

satisfy the hypotheses of Lemma 2.3 and Lemma 2.4. Therefore by Theorem 4.2 the function  $\widehat{u}$  is a solution of the stationary KdV equation. The values  $\widehat{\mu}_i$  are determined by (4.1), the values  $\widehat{a}_i$  are found from (4.4).  $\square$

Thus we have an action of the group  $\mathbb{C}^*$  on the space  $\mathbf{R}_g$ . It is obvious that under this action the initial values  $c_j = u^{(j)}(0)$  are transformed as  $\widehat{c}_j = \kappa^{2j+1} c_j$ .

Denote by  $\widehat{w}$  the  $w$ -function of the solution  $\widehat{u}$ .

**Lemma 10.2.** *The  $w$ -functions  $w$  and  $\widehat{w}$  of the solutions  $u$  and  $\widehat{u}$  are related as follows:*

$$\widehat{w}(t_1, \dots, t_g) = w(\kappa x, \kappa^3 t_2, \dots, \kappa^{2g-1} t_g)$$

**Proof.** The functions  $u_i$ ,  $\widehat{u}_i$  are related by equation (10.1). It follows from (8.1), (9.4) that  $-2\partial_i \partial_j \widehat{w}(t_1, \dots, t_g) = \widehat{p}_{ij} = \kappa^{2i+2j-2} p_{ij} = -2\partial_i \partial_j w(\kappa x, \kappa^3 t_2, \dots, \kappa^{2g-1} t_g)$ . Since the  $w$ -function is unique, this completes the proof.  $\square$

Consider now the  $w$ -function as a function on the space  $\mathbb{C}^g \times \mathbf{R}_g$ .

**Theorem 10.1.** *The function  $w$  satisfies the homogeneity condition:*

$$w(t_1, \dots, t_g, a_0, \dots, a_{g-2}, c_0, \dots, c_{2g}) = w(\kappa t_1, \dots, \kappa^{2g-1} t_g, \kappa^{-2g} a_0, \dots, \kappa^{-4} a_{g-2}, \kappa^{-1} c_0, \dots, \kappa^{-2g-1} c_{2g})$$

**Proof.** The theorem follows directly from Lemma 10.1 and Lemma 10.2.  $\square$

Consider the function  $\hat{\mathbf{u}}(\xi) = \sum_{i=1}^g \hat{u}_i \xi_i$ . It follows from (10.1) that

$$(10.2) \quad \hat{\mathbf{u}}(x, t_2, \dots, t_g; \xi) = \mathbf{u}(\kappa x, \kappa^3 t_2, \dots, \kappa^{2g-1} t_g; \kappa^{-2} \xi).$$

**Theorem 10.2.** *Let  $\Phi(t_1, \dots, t_g; \xi^{-1}, \alpha_1, \dots, \alpha_g)$  be a common eigenfunction of the operators  $\mathcal{L}, \mathcal{U}_1, \dots, \mathcal{U}_g$  with the eigenvalues  $E = \xi^{-1}, \alpha_1, \dots, \alpha_g$  (see Section 7). Then the function*

$$(10.3) \quad \tilde{\Phi}(t_1, \dots, t_g; \xi^{-1}, \alpha_1, \dots, \alpha_g; \kappa) = \Phi(t_1 \kappa, \dots, t_g \kappa^{2g-1}; \xi^{-1} \kappa^{-2}, \alpha_1 \kappa^{-3}, \dots, \alpha_g \kappa^{-2g-1})$$

*is regular as a function of  $\kappa$  in the vicinity of the origin, and*

$$\tilde{\Phi} = \exp \left( \sum_{1 \leq i \leq j \leq g} \alpha_j \xi^{j-i+1} t_i \right) + O(\kappa).$$

**Proof.** Denote  $\hat{t} = (t_1 \kappa, \dots, t_g \kappa^{2g-1})$ . Using (7.9), (7.11), one gets

$$\begin{aligned} \frac{\partial_i \tilde{\Phi}}{\tilde{\Phi}} &= \kappa^{2i-1} (\partial_i \tilde{F})(\hat{t}; \xi^{-1} \kappa^{-2}, \alpha_1 \kappa^{-3}, \dots, \alpha_g \kappa^{-2g-1}) \\ &= \frac{\xi^{1-i}}{4(2 - u_1(\hat{t}) \xi \kappa^2 - \dots)} \left( 2 \sum_{j \geq i} (4\alpha_j \xi^j - u'_j(\hat{t}) \xi^j \kappa^{2j+1}) - \right. \\ &\quad \left. - \sum_{j < i} u_j(\hat{t}) \xi^j \kappa^{2j} \sum_{j \geq i} (4\alpha_j \xi^j - u'_j(\hat{t}) \xi^j \kappa^{2j+1}) + \sum_{j \geq i} u_j(\hat{t}) \xi^j \kappa^{2j} \sum_{j < i} (4\alpha_j \xi^j - u'_j(\hat{t}) \xi^j \kappa^{2j+1}) \right). \end{aligned}$$

Note that  $\hat{t} \rightarrow (0, \dots, 0)$  and  $u_i(\hat{t}) \rightarrow u_i(0)$  as  $\kappa \rightarrow 0$ . Therefore we obtain

$$\frac{\partial_i \tilde{\Phi}}{\tilde{\Phi}} = \sum_{j \geq i} \alpha_j \xi^{j-i+1} + O(\kappa). \quad \square$$

This allows to obtain a deformation of the function  $\Phi$ . We see that  $\tilde{\Phi}$  tends to the function  $\Phi_0$  of (7.14) as  $\kappa \rightarrow 0$ .

Consider the space  $L = \mathbb{C}^g \times \mathbb{C}^* \times \mathbb{C}^g$  with coordinates  $(t_1, \dots, t_g; \xi, \alpha_1, \dots, \alpha_g)$ . Consider also an action of the group  $\mathbb{C}^*$  on the space  $L$  given by a formula  $\kappa(t_1, \dots, t_g; \xi, \alpha_1, \dots, \alpha_g) = (t_1 \kappa, t_2 \kappa^3, \dots, t_g \kappa^{2g-1}; \xi \kappa^2, \alpha_1 \kappa^{-3}, \dots, \alpha_g \kappa^{-2g-1})$ . This defines a projection  $p : L \rightarrow M$  where  $M = L/\mathbb{C}^*$ . Take some small  $\varepsilon > 0$  and denote  $L_\varepsilon = \{(t_1, \dots, t_g; \xi, \alpha_1, \dots, \alpha_g) \in L : |\xi| \geq \varepsilon\}$  and  $\partial L_\varepsilon = \{(t_1, \dots, t_g; \xi, \alpha_1, \dots, \alpha_g) \in L : |\xi| = \varepsilon\}$ . Glue the boundary  $\partial L_\varepsilon$  to the space  $M$  using the projection  $p$  to obtains the space  $Z_\varepsilon = L_\varepsilon \cup_p M$ .

Let  $\varepsilon_2 < \varepsilon$ . Then there is a map  $L_\varepsilon \rightarrow L_{\varepsilon_2}$  defined by the formula  $(t_1, \dots, t_g; \xi, \alpha_1, \dots, \alpha_g) \rightarrow \kappa(t_1, \dots, t_g; \xi, \alpha_1, \dots, \alpha_g)$  where  $\kappa = \varepsilon^{-1/2} \varepsilon_2^{1/2}$ . This map sends the boundary  $\partial L_\varepsilon$  to the boundary  $\partial L_{\varepsilon_2}$ , so it can be lifted to a map  $Z_\varepsilon = L_\varepsilon \cup M \rightarrow Z_{\varepsilon_2} = L_{\varepsilon_2} \cup M$ . Denote  $Z = \lim_{\varepsilon \rightarrow 0} Z_\varepsilon$  and recall that  $\mathbb{C}^* \times V^* \subset L$ .

Consider the embedding  $\mathbb{C}^g \times V^* \rightarrow Z$ . This embedding covers the embedding  $\Gamma^* \rightarrow \Gamma$ . Approaching the limit point in  $\Gamma$  corresponds to  $\xi \rightarrow 0$  is the space  $M \subset Z$ . So we get the following result:

**Theorem 10.3.** *On the space  $Z$  there is a function  $\widehat{\Phi}$ , such that  $\widehat{\Phi}|_L = \Phi$  and  $\widehat{\Phi}|_M = \Phi_0$ .*

If  $\xi \rightarrow 0$  then for the restriction  $\Phi|_\Gamma = \Phi(t_1, \dots, t_g, \xi^{-1}, 0, 0, \dots, \alpha_g)$  one has  $\Phi \sim \exp\left(\sum_{1 \leq j \leq g} \alpha_g \xi^{g-j+1} t_j\right)$ . Take a local parameter  $k = \alpha_g \xi^g$ . It follows now from the equation  $(\alpha_g \xi^g)^2 = \mu(\xi) = 4\xi^{-1} + O(\xi)$  that  $\Phi \sim \exp(\sum_{j=1}^g k^{2j-1} t_j)$ .

So, the restriction  $\Phi|_\Gamma$  has the same analytic properties as the Baker–Akhiezer function ([16]) of the solution  $u$ . By the uniqueness of the Baker–Akhiezer function we conclude that  $\Phi|_\Gamma$  coincides with the Baker–Akhiezer function.

## 11. EXAMPLES

In this section we demonstrate the key constructions of the paper in the cases  $g = 1$  and  $g = 2$ .

11.1.  $g = 1$ . We start with a solution  $u$  of the classical stationary KdV equation  $u''' - 6uu' = 0$ . Suppose that  $x = 0$  is a regular point of the function  $u$ . Then the solution  $u$  with the given values  $c_0 = u(0)$ ,  $c_1 = u'(0)$ ,  $c_2 = u''(0)$  is unique in a neighbourhood of the point  $x = 0$ .

The key equation (4.1) becomes

$$4(4\xi^{-1} + \mu_1\xi + \mu_2\xi^2) = (u')^2\xi^2 + 2u''\xi(2 - u\xi) + 4(\xi^{-1} + u)(2 - u\xi)^2,$$

hence  $\mu_1 = u'' - 3u^2$ ,  $\mu_2 = 1/4((u')^2 - 2u''u + 4u^3)$ . It is easy to see that  $\mu'_1 = 0$  and  $\mu'_2 = 0$ . Therefore  $\mu_1, \mu_2$  are constants and so  $\mu_1 = c_2 - 3c_0^2$ ,  $\mu_2 = 1/4(c_1^2 - 2c_2c_0 + 4c_0^3)$ .

The equation of the hyperelliptic curve is

$$4(4\xi^{-1} + \mu_1\xi + \mu_2\xi^2) = y^2.$$

The  $w$ -function is  $w = \exp(-\phi(x))$  where  $\phi(x) = \frac{1}{2} \int_0^x \int_0^x u(x) dx$ .

The birational equivalence  $v : \mathbf{R}_1 \rightarrow \mathcal{U}_1$  is given by the formula  $v(u_1) = (\Gamma, (\xi, y))$ , where  $\xi = 2/c_0$ ,  $y = 2c_1/c_0$ .

Let  $\Phi$  be a common eigenfunction of the operators  $\mathcal{L}$  and  $\mathcal{U}_1 = A_1$  with the eigenvalues  $E = \xi^{-1}$  and  $\alpha$ , respectively. Then the logarithmic derivative of the function  $\Phi$  is

$$\frac{\Phi'}{\Phi} = \frac{4\alpha\xi - u'(x)\xi}{2(2 - u(x)\xi)}$$

where  $4\alpha\xi = 2\sqrt{4\xi^{-1} + \mu_1\xi + \mu_2\xi^2}$ . Therefore  $\partial_x \log \Phi \sim \xi^{-1/2}$  as  $\xi \rightarrow 0$ . Let  $z \in \mathbb{C}$  be such that  $u(z) = 2\xi^{-1}$  and  $u'(z) = 4\alpha$ . Then

$$\widetilde{F}'(x; \xi^{-1}, \alpha) = \widetilde{F}'(x; z) = \frac{1}{2} \frac{u'(z) - u'(x)}{u(z) - u(x)}.$$

11.2.  $g = 2$ . It follows from the equation  $[\mathcal{L}, A_2 + a_0 A_0] = 0$  that  $u^{(5)} - 10uu^{(3)} - 20u''u' + 30u^2u' + 16a_0u' = 0$ . Equation (4.4) implies that  $\mu_1 = 8a_0$ . We have  $\mathbf{u}(\xi) = u\xi + u_2\xi^2$ . The equation (4.1) gives

$$\begin{aligned} & 4(4\xi^{-1} + \mu_1\xi + \mu_2\xi^2 + \mu_3\xi^3 + \mu_4\xi^4) \\ &= \xi(-12u^2 - 16u_2 + 4u'') + \xi^2(4u^3 - 8uu_2 + (u')^2 - 2uu'' + 4u_2'') \\ &+ \xi^3(8u^2u_2 + 4u_2'^2 + 2u'u_2' - 2u_2u'' - 2uu_2'') + \xi^4(4uu_2'^2 + (u_2')^2 - 2u_2u_2''). \end{aligned}$$

Therefore,  $u_2 = \frac{1}{4}(u'' - 3u^2 - 8a_0)$ . Now we can describe  $\mu_2$ ,  $\mu_3$  and  $\mu_4$  as constants in the following ordinary differential equation for  $u$ :  $\mu_2 = \frac{1}{4}(4u^{(4)} - 10uu'' - 5(u')^2 + 10u^3 + 16a_0u)$ ,  $\mu_3 = \frac{1}{16}(2u'u''' - 2uu^{(4)} - 2(u'')^2 - 15u^4 + 8u^2u'' - 16u^2a_0 + 12u^2u' + 64a_0)$ ,  $\mu_4 = \frac{1}{64}((u''')^2 + 16(u'')^2u - 2u''u^{(4)} + 12(u')^2u'' + 6u^2u^{(4)} + 32u^5 - 30u''u^3 - 12u'''u'u - 160a_0uu'' + 132a_0u^3 + 16a_0u^{(4)} - 96a_0(u')^2 + 256a_0^2u)$ . At last, the birational equivalence  $v : \mathbf{R}_2 \rightarrow \mathcal{U}_2$  is given by the formula  $v(u_1) = v(c_0, \dots, c_4, a_0) = (\Gamma, [(\xi_1, y_1), (\xi_2, y_2)])$ . Here for the construction of  $\Gamma$  the coefficients  $\mu_1, \mu_2, \mu_3, \mu_4$  are used, obtained from the formula above by substitution  $c_k$  for  $u^{(k)}$ . The pairs  $(\xi_i, y_i)$  are the following ones:  $\xi_1, \xi_2$  are the roots of the equation  $2 - c_0\xi - \frac{1}{4}(c_2 - 3c_0^2 - 8a_0)\xi^2 = 0$  and  $y_1, y_2$  are defined by the formula  $y_i = c_1\xi_i + \frac{1}{4}(c_3 - 6c_1c_0)$ .

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#### REFERENCES

- [1] M.Adler, J.Moser "On a class of Polynomials connected with the Korteweg- de Vries equation", Commun.Math.Phys., 61,(1978), 1 – 30.
- [2] V.M. Buchstaber, V.Z. Enolskii, D.V. Leykin, "Hyperelliptic Kleinian functions and applications", Advances in Math. Sciences, AMS Translations, Series 2, Vol. 179 (1997), 1 – 33.
- [3] V.M. Buchstaber, V.Z. Enolskii, D.V. Leykin, "Kleinian functions, hyperelliptic Jacobians and applications", Review in Mathematics and Mathematical Physics, vol. 10:2, Gordon and Breach, (1997), 1 – 125.
- [4] V.M. Buchstaber, V.Z. Enolskii, D.V. Leykin, Rational analogues of Abelian functions, Funct.Anal.Appl. 33:2 (1999), 1–15
- [5] V.M. Buchstaber, S.Yu. Shorina, Commuting multidimensional differential operators of order 3 generating the KdV hierarchy, Russ. Math. Surv. 58:3 (2003), 187–188. (russian pages)
- [6] V.M. Buchstaber, S.Yu. Shorina, The  $w$ -function of the KdV hierarchy, Russ. Math. Surv. 58:5 (2003), 145 – 146. (russian pages)
- [7] J.L. Burchnell, T.W. Chaundy, Commutative ordinary differential operators, Proc. London Math. Soc. Ser. 2 21 (1923), 420 – 440.
- [8] L.A.Dikii, I.M Gelfand, Asymptotic behaviour of the resolvent of Sturm-Liouville equations, and the algebra of the Korteweg-de Vries equations, Russian Math. Surveys 30:5 (1975), 77 – 100.
- [9] L.A.Dikii, I.M Gelfand, Integrable nonlinear equations and Liouville theorem, Func. Anal. Appl. 13(1) (1979), 8 – 20.
- [10] B. A. Dubrovin, Periodic problems for the Korteweg-de Vries equation in the class of finite band potentials, Funct. Anal. Appl. 9 (1975), 215 – 223.
- [11] B. A. Dubrovin "Theta functions and nonlinear equations." Russian Math. Surv. 36 (1981), 11 – 80.
- [12] B. A. Dubrovin, V.B. Matveev, S.P. Novikov, Nonlinear equations of KdV type, finite-zone linear operators, and Abelian varieties Russ. Math. Surv. 31:1 (1976), 59 – 146.
- [13] R. Hirota, "Direct methods in soliton theory", Topics in Current Physics, 17 (1980), 157 – 175.
- [14] A. R. Its, V. B. Matveev, "Hill s operator with finitely many gaps" Func. Anal. Appl. 9 (1975), 69 – 70.
- [15] I.M. Krichever, Integration of nonlinear equations by the methods of algebraic geometry, Funct. Anal. Appl. 11 :1 (1977), 12–26.
- [16] I.M.Krichever , Methods of algebraic geometry in the theory of non-linear equations, Russ. Math. Surv., 32:6, (1977), 183 – 208.
- [17] I. M.Krichever, S. P.Novikov, "Holomorphic Bundles over Algebraic Curves, and Nonlinear Equations." Russ. Math. Surv. 35, (1980), 53 – 80.
- [18] P.D. Lax, "Integrals of nonlinear equations of evolution and solitary waves", Communications on Pure and Applied Mathematics, 21 (1968), 467 – 490.



- [19] S.V. Manakov, S.P. Novikov, L.P. Pitaevskii, V.E. Zakharov, "Theory of Solitons: The Inverse Scattering Method", Consultants Bureau, New York (1984)
- [20] S.P. Novikov, The periodic problem for the Korteweg-de Vries equation, *Funct. Anal. Appl.* 8 (1975), 236 -246.